16.002 Lecture 18

Stability of Feedback Control Systems

May 9, 2008

Today's Topics

Stabilizing an unstable system
Stability evaluation using frequency responses

Take Away

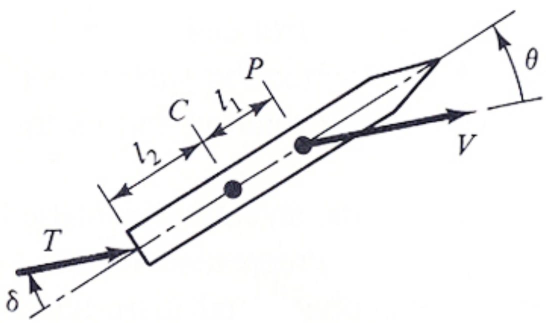
Feedback systems stability can be evaluated using system frequency response contours

Required Reading

O&W-11.1, 11.2.1, 11.2.2, 11.2.3 (except discrete time systems), 11.2.6

As we discussed last time, feedback control systems provide a means for significantly enhancing the performance of all kinds of systems. Furthermore, since aerospace vehicles commonly exhibit unsatisfactory or even unstable dynamics, feedback control systems are a necessary and inherent part of almost all modern aerospace systems.

To begin today we are going to study an unstable system and demonstrate how to make it stable. Our model will be the rocket that we studied in Lecture 13. We had the following diagram



(a)

and we linearized the dynamics and developed the transfer function relating thrust angle $\delta(t)$ to the vehicle angle of attack $\theta(t)$. The transfer function is

$$H(s) = \frac{(T l_2 / J)}{s^2 - \left(\frac{C_n l_1}{J}\right)}$$

where the various constants are

T=thrust
l_1=distance from the center of mass (C) to the center of pressure (P)
l_2=distance from the center of mass (C) to the rocket engine
J=Moment of inertia of the rocket vehicle about C
C_n=linearized aerodynamic constant relating pitch moment to angle of attack

We will assume the following parameter values

$$\frac{T\ell_1}{J} = 1.0 \qquad \frac{C_n \ell_1}{J} = 1.0$$

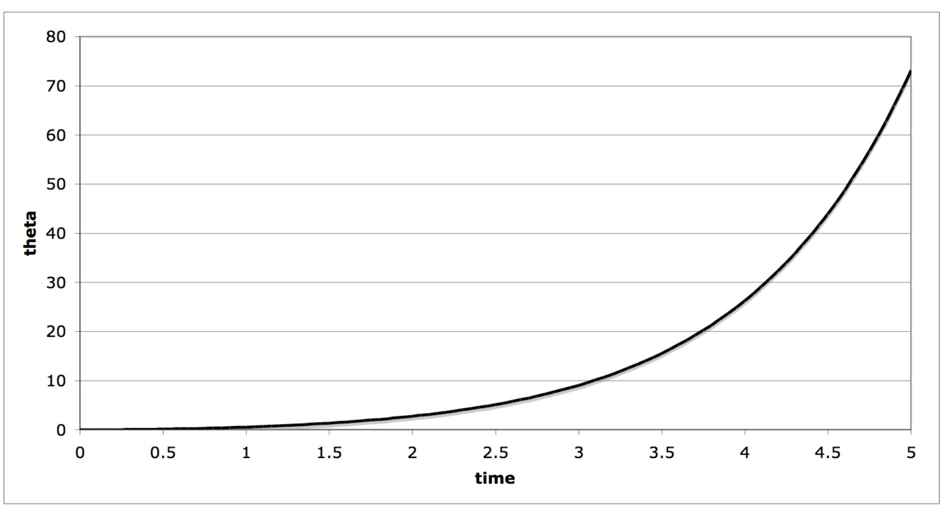
so the system transfer function is

$$H_R(s) = \frac{1}{s^2 - 1}$$

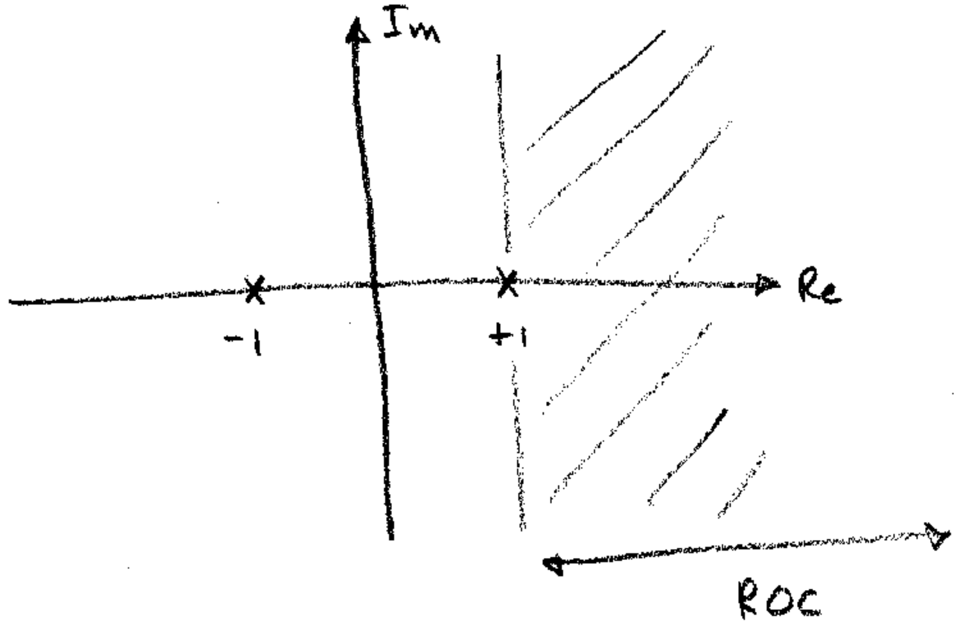
and the system has poles at

$$s_p = \pm 1$$

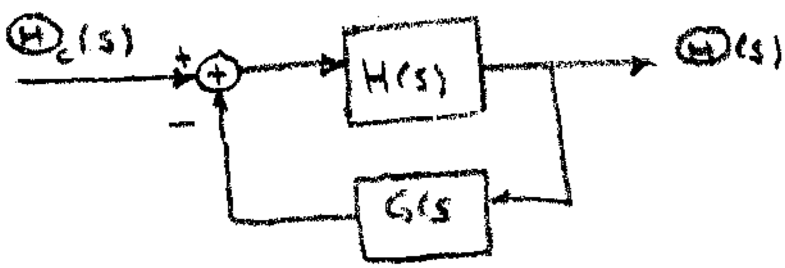
As we discussed in Lecture 14, this system is unstable because of the pole at +1.0. Its response to a step input looks like this



The system pole/zero diagram is



Our goal is to create a feedback system that will stabilize the rocket. The feedback system can be represented as follows

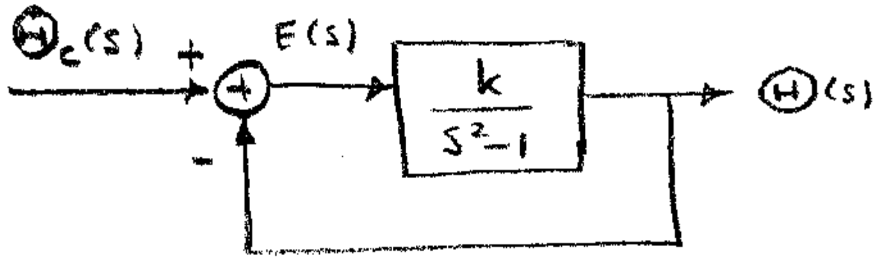


where $\Theta_c(s)$ is the L.T. of the input commanded angle of attack, $\Theta(s)$ is the L.T. of the closed loop system output angle of attack, *H(s)* is the aggregate transfer function of all elements in the forward path and *G(s)* is the aggregate transfer function of all elements in the feedback path.

First we will try the simplest form of feedback loop closure to stabilize this system, which is unity feedback and a forward loop gain *K*. Hence the forward and feedback path transfer functions are

$$H(s) = \frac{k}{s^2 - 1} \qquad G(s) = 1$$

and the closed loop feedback system diagram for the rocket becomes



Our goal is to design the feedback controller so that the vehicle will follow the input commands as closely as possible. The feedback signal is obtained from an inertial system onboard the rocket that holds a constant reference with respect to inertial space. Its output, which is the feedback signal, measures the rocket angle of attack relative to that reference. The difference between the commanded angle of attack and the feedback angle, from the inertial system, is computed within the control system to produce the error signal *E(s)*. The error signal is input the controller which, in this case, is the constant gain *K*.

Last time we showed that the general form for a closed loop system transfer function is

$$H_{CL}(s) = \frac{H(s)}{1 + G(s)H(s)}$$

and substituting from above

$$H_{CL}(s) = \frac{\left(\frac{k}{s^2-1}\right)}{1+\left(\frac{k}{s^2-1}\right)} = \frac{k}{s^2+(k-1)}$$

The denominator is a second order polynomial so the poles of the closed loop system are readily determined as

$$S_p = \pm\sqrt{k-1}$$

For small positive values of the gain *K* the poles are both real and symmetrically placed on each side of the origin, one to the left and one to the right of the origin. The closed loop system response is similar to what we showed earlier for the rocket alone.

The system is still unstable!

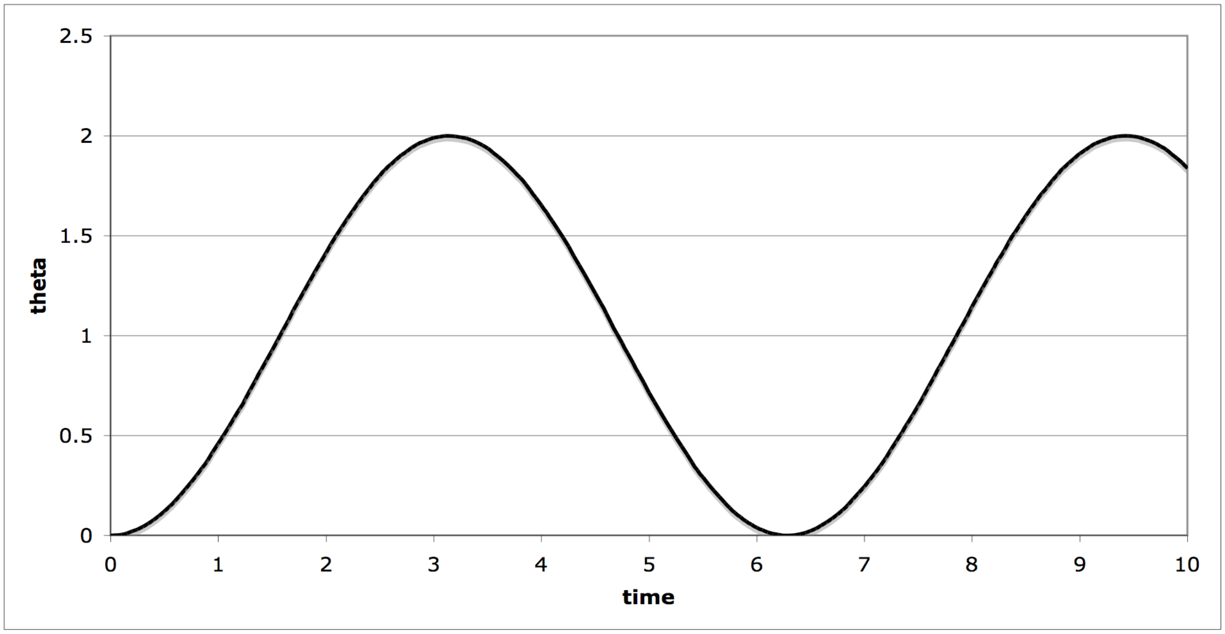
As we increase *K* the poles both move toward the origin until, at *K=1.0* both poles are exactly at the origin and the closed loop transfer function is

$$H_{CL}(s) = \frac{1}{s^2} \qquad k=1$$

The system is now a double integrator so a step input command will result in a parabolic departure, similar to what we saw above.

The system is still unstable!

If the gain increases further then the poles become imaginary and move symmetrically up and down the imaginary axis. The closed loop system is just at the edge of instability and its response to any input will be oscillatory, at the undamped natural frequency, which is the pole location on the imaginary axis. Now he closed loop system response to a step input looks like this.



Performance is still unsatisfactory!

In order to stabilize this system, and make it perform properly, we need a different kind of feedback. In this case we will use a standard control system method, called a proportional plus derivative feedback compensator. The transfer function for the feedback path is now

$$G(s) = cs + 1$$

So the proportional gain in the feedback path is the same as before but we have added another term, which is the constant *c* times the derivative of the angle of attack. Recall that the general closed loop system transfer function is

$$H_{CL}(s) = \frac{H(s)}{1 + G(s)H(s)}$$

and if we substitute *G(S)* and *H(s)* into this equation then

$$H_{CL}(s) = \frac{\left(\frac{k}{s^2 - 1}\right)}{1 + (cs+1)\left(\frac{k}{s^2 - 1}\right)}$$

$$= \frac{k}{s^2 + cks + (k-1)} = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

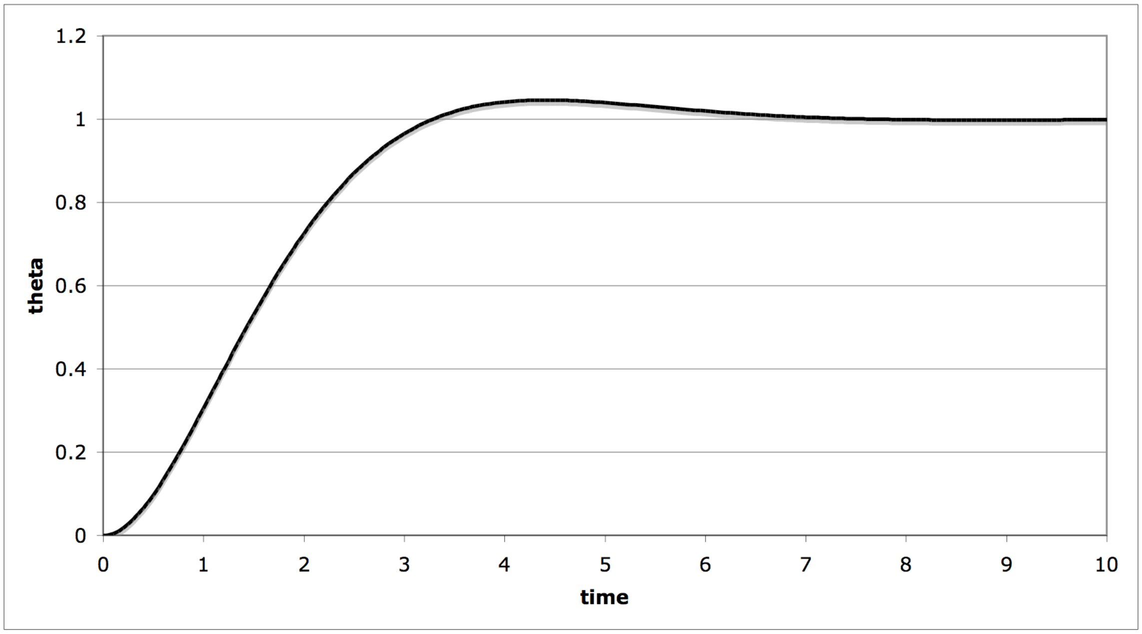
The denominator is now in the familiar quadratic form and we can identify

$$\omega_n = (k-1) \qquad 2\zeta\omega_n = ck$$

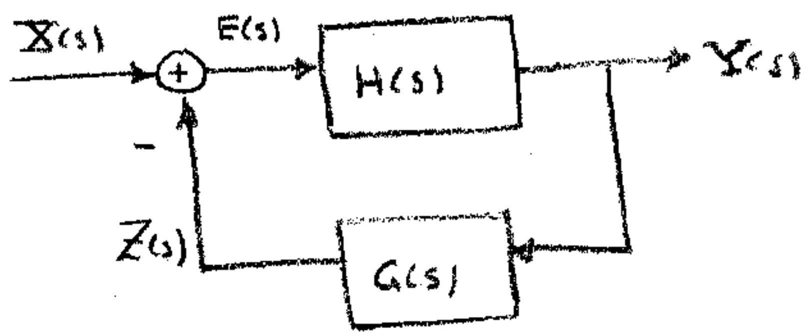
By appropriate choice of the constants *K* and *c* we can achieve any desired undamped natural frequency and damping ratio for the closed loop system. We choose an undamped natural frequency of 1.0 and a damping ratio of 0.7, so the two parameter values *K* and *c* are immediately determined as.

$$k = 2 \qquad c = 0.7$$

The closed loop system response to a step input now looks like this.



The general Feedback Control System Design Problem
Consider the general situation for feedback as shown here



We wish to study the stability of feedback controls so we will explore the situation when such a system is just at the very edge of instability. In particular, we assume that *G(s)* and *H(s)* are such that the closed loop system is at the very edge of instability. In this situation one or more of the system poles is precisely on the imaginary axis. The typical case is when there are two complex conjugate poles on the imaginary axis. We saw this situation for the rocket when we did not have the proportional plus derivative feedback compensator and the gain *K* was larger than one. The response to a step input was oscillatory.

Let's look more closely at the closed loop system operation in such a situation. Assume that the input is zero so

$$X(s) = 0$$

and that the system has been perturbed. Since it is just at the edge of instability it will oscillate at constant amplitude and at a resonant frequency determined by the location of the closed loop poles on the imaginary axis. In steady state every signal in the loop will be a sinusoid at that same frequency.

If there is no input then the error signal is the negative of the feedback signal so

$$E(s) = -Z(s) = -G(s)H(s)E(s)$$

and since the system is in steady state oscillation we must have

$$G(j\omega)H(j\omega) = -1$$

where the frequency ω is the resonant frequency of oscillation.

We can also arrive at this same result in somewhat simpler fashion by examining the general closed loop system transfer function

$$H_{CL}(s) = \frac{H(s)}{1 + G(s)H(s)}$$

By definition, a pole of the system occurs when

$$H_{CL}(s) \longrightarrow \infty$$

which occurs when

$$1 + G(s)H(s) = 0$$

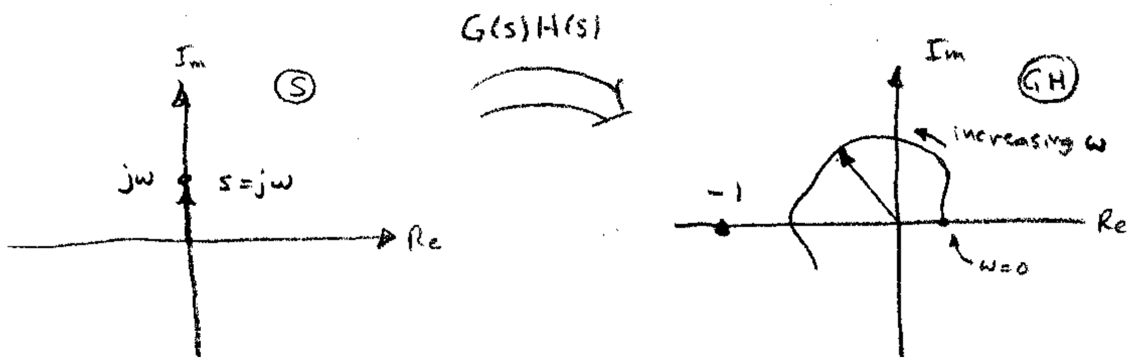
and if the pole is just at the point of instability it will lie on the imaginary axis, so our condition becomes

$$G(j\omega)H(j\omega) = -1$$

which is the same result as earlier.

The product *G(s)H(s)* is called the loop transfer function because it is the transfer function that characterizes how signals are affected as they pass through the loop. What we have learned is that if the closed loop system is just at the point of instability, so that at least one of its poles lies on the imaginary axis at the point $j\omega$, then at that pole the frequency response of the loop transfer function $G(j\omega)H(j\omega)$ must equal -1.0, for some value of ω. We will use this fact to analyze the stability of feedback control systems.

Our approach will be to create a diagram that represents the function $G(j\omega)H(j\omega)$ for all points $j\omega$ on the imaginary axis. Recall that such a function transforms each point in the s plane to a corresponding point in the *GH* plane. Conceptually it looks like this



As the frequency ω goes for zero to +infinity the function $G(j\omega)H(j\omega)$ will create a contour in the *GH* plane. We will then look for that contour to pass through the -1.0 point to determine stability. This diagram is called a Nyquist plot.

<u>Examples:</u>