Wing Bending Calculations
Lab 11 Lecture Notes

Nomenclature

y	spanwise coordinate	η	normalized spanwise coordinate ($= 2y/b$)
q	net beam loading	c	local wing chord
$\mathcal{S}$	shear	S_{wing}	wing area
M	bending moment	b	wing span
θ	deflection angle ($= dw/dx$)	λ	taper ratio
w	deflection	E	Young's modulus
κ	local beam curvature	δ	tip deflection
L'	lift/span distribution	N	load factor
m'	wing mass/span distribution	L	lift
I	bending inertia	W	weight
i	spanwise station index	g	gravitational acceleration
n	last station index at tip	$()_o$	quantity at wing root

Loading and Deflection Relations

The net wing beam load distribution along the span is given by

$$q(y) \;=\; L'(y) \,-\, N\,g\,m'(y) \tag{1}$$

where $m'(y)$ is the local mass/span of the wing, and N is the load factor. In steady level flight we have $N=1$. The net loading $q(y)$ produces shear $\mathcal{S}(y)$ and bending moment $M(y)$ in the beam structure. This resultant distribution produces a deflection angle $\theta(y)$, and deflection $w(y)$ of the beam, as sketched in Figure 1.

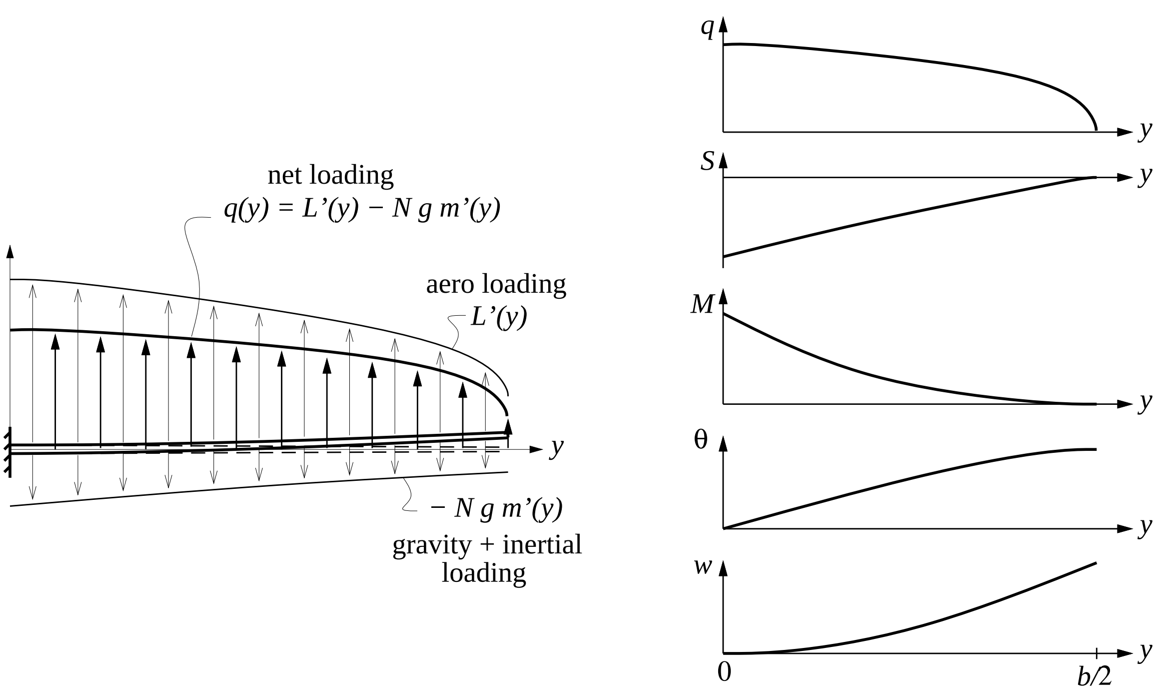

Figure 1: Aerodynamic and mass loadings, and resulting structural loads and deflection.

These variables are related by differential equations derived via simple Bernoulli-Euler beam model.

$$
\begin{aligned}
\frac{d\mathcal{S}}{dy} &= q && (2)\\
\frac{dM}{dy} &= \mathcal{S} && (3)\\
\frac{d\theta}{dy} &= \frac{M}{EI} && (4)\\
\frac{dw}{dy} &= \theta && (5)
\end{aligned}
$$

For a cantilevered wing beam, these equations have the following boundary conditions.

$$
\begin{aligned}
y = b/2: &\quad \mathcal{S} = 0 && (6)\\
y = b/2: &\quad M = 0 && (7)\\
y = 0: &\quad \theta = 0 && (8)\\
y = 0: &\quad w = 0 && (9)
\end{aligned}
$$

For some given loading $q(y)$ and bending stiffness $EI(y)$ distributions along the wing, and the four boundary conditions (6)–(9), the four equations (2)–(5) can integrated, numerically if necessary. One result of interest from this integration is the deflection distribution $w(y)$, or just the tip deflection $\delta \equiv w(b/2)$.

Simplified Load Distribution

The lift distribution $L'(y)$ needed to define $q(y)$ depends on the induced angle $\alpha_i(y)$ and hence the overall wing shape in a complicated manner. One reasonable simplification is to assume that the net aerodynamic + weight loading in equation (1) is proportional to the local chord,

$$q(y) \simeq K_q \, c(y) \qquad (10)$$

as shown in Figure 2.

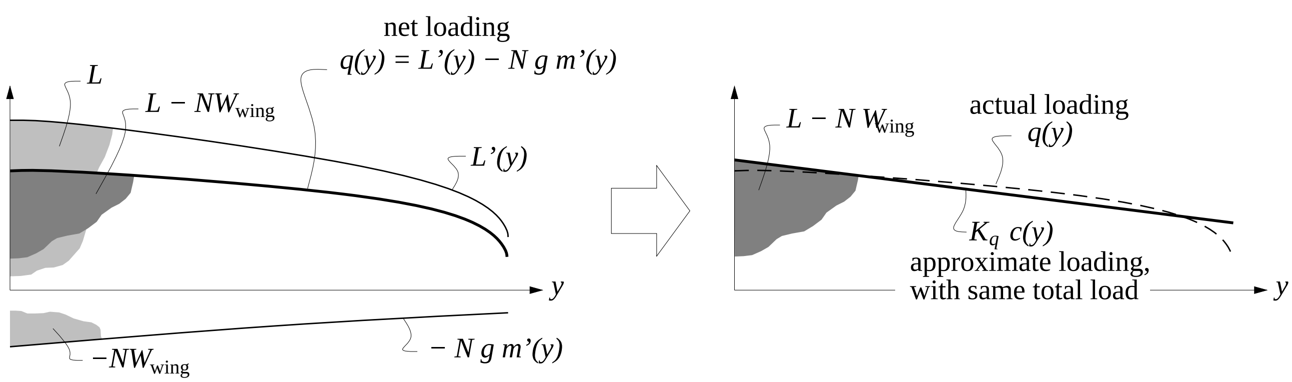

Figure 2: Net loading assumed to scale as local chord $c(y)$.

This is equivalent to assuming a constant local $c_\ell = C_L$, and that the local wing mass distribution $m'(y)$ scales as the chord. The constant K_q is best set such that the approximate and actual loadings have the same total integrated loads.

$$\int_{-b/2}^{b/2} K_q \, c \, dy \;=\; \int_{-b/2}^{b/2} (L' \,-\, Ngm') \; dy \tag{11}$$

$$K_q \, S_{\text{wing}} \;=\; L \,-\, NW_{\text{wing}} \tag{12}$$

$$K_q \;=\; \frac{L - NW_{\text{wing}}}{S_{\text{wing}}} \;=\; \frac{NW_{\text{cent}}}{S_{\text{wing}}} \tag{13}$$

where W_{cent} is the central weight, of everything except the wing.

Simplified Deflection Calculations

For preliminary or optimization work, numerical integration of the beam equations (2)–(5) of each candidate wing is unwieldy. For estimation, an effective approximation is to assume that the beam curvature

$$\kappa(y) \;\equiv\; \frac{d^2 w}{dy^2} \;=\; \frac{d\theta}{dy} \;=\; \frac{M(y)}{EI(y)}$$

is constant, and taken from some representative location such as the wing root at $y\!=\!0$.

$$\kappa(y) \;\simeq\; \kappa_0 \;=\; \frac{M_0}{EI_0} \tag{14}$$

$$\theta(y) \;=\; \int_0^y \kappa_0 \, dy \;=\; \kappa_0 \, y \tag{15}$$

$$w(y) \;=\; \int_0^y \theta \, dy \;=\; \frac{1}{2} \kappa_0 \, y^2 \tag{16}$$

For a straight-taper wing with taper ratio $c_t/c_r\!=\!\lambda$, the chord distribution is

$$c(y) \;=\; \frac{S_{\text{wing}}}{b} \frac{2}{1+\lambda} \left[1 + (\lambda - 1) \frac{2y}{b} \right] \tag{17}$$

and the corresponding approximate loading is then given by (10), and by the K_q definition (13).

$$q(y) \;\simeq\; K_q \, c(y) \;=\; \frac{NW_{\text{cent}}}{b} \frac{2}{1+\lambda} \Big[1 + (\lambda - 1)\eta \Big] \tag{18}$$

$$\eta \;\equiv\; \frac{2y}{b} \tag{19}$$

To simplify the subsequent integrations, the y coordinate has been replaced with the equivalent and more convenient normalized coordinate η, which runs $\eta = 0 \ldots 1$ root to tip. The shear and bending moment are then calculated by integrating equations (2) and (3).

$$\mathcal{S}(y) \;=\; \int_y^{b/2} q(y) \; dy \tag{20}$$

$$\mathcal{S}(\eta) \;=\; \frac{b}{2} \int_\eta^1 q(\eta) \; d\eta \tag{21}$$

$$\;=\; \frac{NW_{\text{cent}}}{b} \frac{2}{1+\lambda} \frac{b}{2} \int_\eta^1 \Big[1 + (\lambda - 1)\eta \Big] \; d\eta \tag{22}$$

$$= \quad \frac{NW_{\rm cent}}{b}\frac{2}{1+\lambda}\frac{b}{2}\left[1-\eta+(\lambda-1)\frac{1}{2}(1-\eta^2)\right] \tag{23}$$

$$M(y) \quad = \quad \int_y^{b/2} \mathcal{S}(y)\; dy \tag{24}$$

$$M(\eta) \quad = \quad \frac{b}{2}\int_\eta^1 \mathcal{S}(\eta)\; d\eta \tag{25}$$

$$= \quad \frac{NW_{\rm cent}}{b}\frac{2}{1+\lambda}\frac{b^2}{4}\int_\eta^1 \left[1-\eta+(\lambda-1)\frac{1}{2}(1-\eta^2)\right]\; d\eta \tag{26}$$

$$= \quad \frac{NW_{\rm cent}}{b}\frac{2}{1+\lambda}\frac{b^2}{4}\left[1-\eta-\frac{1}{2}\left(1-\eta^2\right)+(\lambda-1)\frac{1}{2}\left(1-\eta-\frac{1}{3}\left(1-\eta^3\right)\right)\right] \tag{27}$$

The root moment is then

$$M_0 \;\equiv\; M(0) \quad = \quad NW_{\rm cent}\,\frac{b}{12}\,\frac{1+2\lambda}{1+\lambda} \tag{28}$$

which is subsequently combined with (14) and (16) to get the following estimate of the tip deflection.

$$\delta \;\equiv\; w(b/2) \quad \simeq \quad \frac{1}{2}\,\kappa_0\left(\frac{b}{2}\right)^2 \quad = \quad \frac{M_0}{EI_0}\,\frac{b^2}{8} \quad = \quad \frac{NW_{\rm cent}}{EI_0}\,\frac{b^3}{96}\,\frac{1+2\lambda}{1+\lambda} \tag{29}$$

Figure 3 shows $\kappa(y)$ for three taper ratios for a solid wing, for which the stiffness varies as

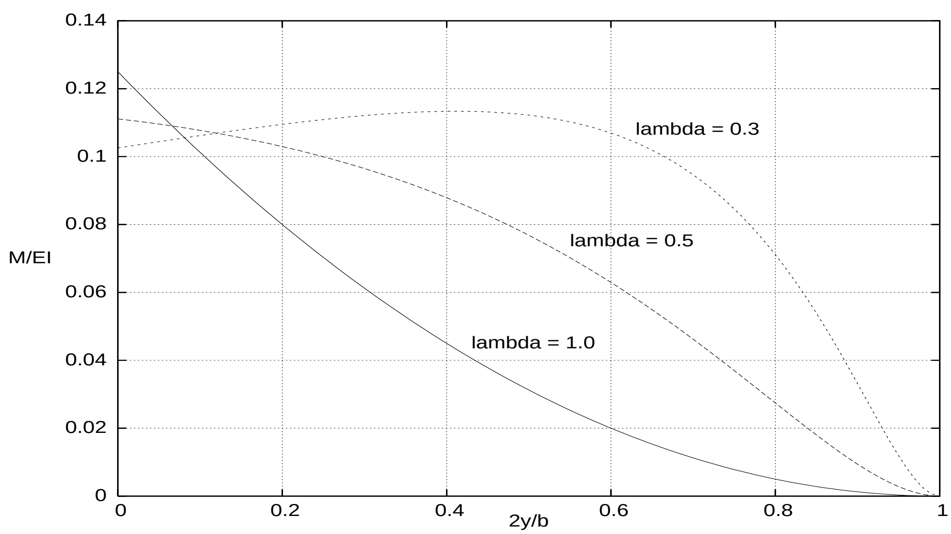

Figure 3: Spanwise distribution of curvature $\kappa(y) = M/EI$ for three taper ratios.

$EI(y) \sim c(y)^4$. It can be seen that the constant $\kappa = \kappa_0$ assumption is poor for a rectangular wing, but reasonable for wings of moderate to strong taper of $\lambda = 0.3 \ldots 0.5$, at least over the inner parts of the wing where the curvature dominates the tip deflection. Figure 4 compares the approximate deflections defined by (16) with the exact deflections, for the three taper ratios. For the untapered $\lambda = 1.0$ wing, the approximation considerably overestimates the tip deflection, but the tapered cases are quite reasonable.

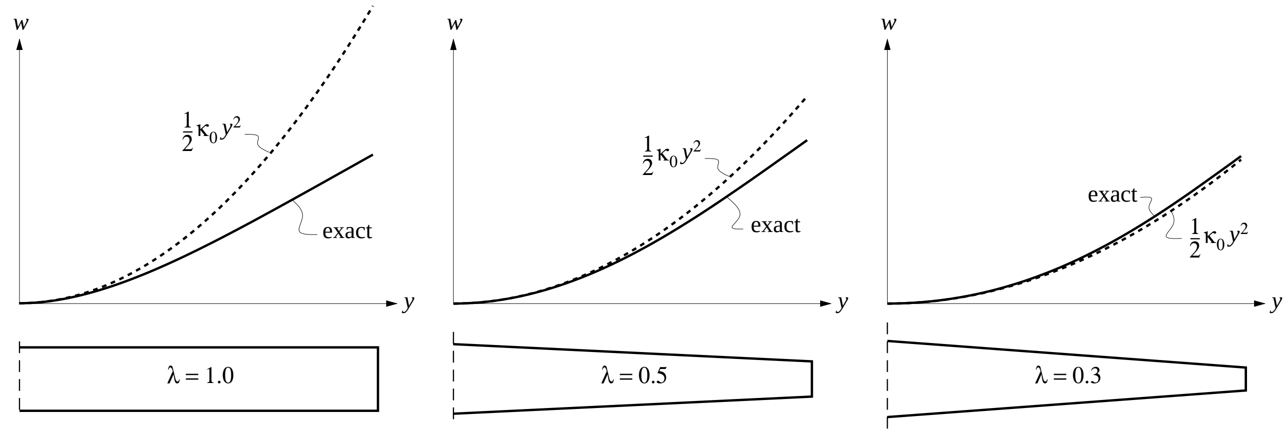

Figure 4: Approximate and exact deflections for three taper ratios.

Area and Bending Inertia of Airfoil Sections

As described above, calculation of the deflection of a wing requires knowing the spanwise bending stiffness distribution $EI(y)$ along the primary axis of loading. For a wing made of a uniform solid material, the modulus E is a simple scaling factor. The moment of inertia $I(y)$ of the airfoil cross-sections about the bending axis x (called the *bending inertia*), is then related only to the airfoil shape given by the upper and lower surfaces $Z_u(x)$ and $Z_\ell(x)$. As shown in Figure 5, both the area A and the total bending inertia I are the integrated contributions of all the infinitesimal rectangular sections, each dx wide and $Z_u - Z_\ell$ tall. The inertia of each such section is appropriately taken about the *neutral surface* position $\bar{z}$ defined for the entire cross section.

$$A = \int_0^c \left[Z_u - Z_\ell \right] \, dx \tag{30}$$

$$\bar{z} = \frac{1}{A} \int_0^c \frac{1}{2} \left[Z_u^2 - Z_\ell^2 \right] \, dx \tag{31}$$

$$I = \int_0^c \frac{1}{3} \left[(Z_u - \bar{z})^3 - (Z_\ell - \bar{z})^3 \right] \, dx \tag{32}$$

These relations assume that the bending deflection will occur in the z direction, which is a good assumption if the x axis is parallel to the airfoil's chord line.

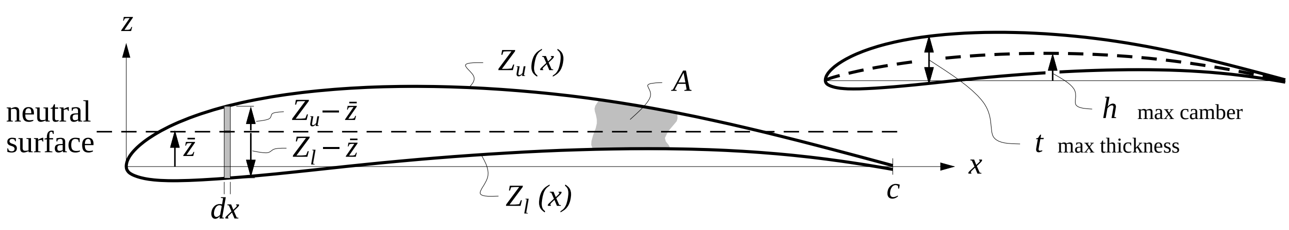

Figure 5: Quantities for determining and estimating the bending inertia of an airfoil section.

Although equations (30) – (32) can be numerically evaluated for any given airfoil (e.g. using XFOIL's `BEND` command), this is unnecessarily cumbersome for preliminary design work, where both A and I are needed for possibly a very large number of candidate airfoils or wings.

For the purpose of approximating A and I, we first define the maximum thickness t, and maximum camber h, in terms of the upper and lower surface shapes. We also define the corresponding thickness and camber ratios τ and ε.

$$t = \max\{\, Z_u(x) - Z_\ell(x) \,\} \tag{33}$$

$$h = \max\{[Z_u(x) + Z_\ell(x)]/2\} \tag{34}$$

$$\tau \equiv t/c$$

$$\varepsilon \equiv h/c$$

Examination of equation (30) indicates that A is proportional to $t\,c$, and examination of (32) indicates that I is proportional to $c\,t(t^2 + h^2)$. This suggests estimating A and I with the following approximations.

$$A \simeq K_A\, c\, t = K_A\, c^2\, \tau \tag{35}$$

$$I \simeq K_I\, c\, t\, (t^2 + h^2) = K_I\, c^4\, \tau(\tau^2 + \varepsilon^2) \tag{36}$$

The proportionality coefficient can be evaluated by equating the exact and approximate A and I expressions above, e.g.

$$K_A \leftarrow \frac{1}{c^2\,\tau} \int_0^c \Big[Z_u - Z_\ell \Big]\; dx \tag{37}$$

$$K_I \leftarrow \frac{1}{c^4\,\tau(\tau^2 + \varepsilon^2)} \int_0^c \frac{1}{3} \Big[(Z_u - \bar{z})^3 - (Z_\ell - \bar{z})^3 \Big]\; dx \tag{38}$$

Evaluating these expressions produces nearly the same K_A and K_I values for most common airfoils:

$$K_A \simeq 0.60 \tag{39}$$

$$K_I \simeq 0.036 \tag{40}$$

Therefore, the very simple approximate equations (35) and (36), with K_A and K_I assumed fixed, are surprisingly accurate. Hence, they are clearly preferred for preliminary design work over the exact but cumbersome equations (30), (31), (32).

Tip deflection of solid-airfoil wing

The root chord of a simple-taper wing is given by equation (17) at $y = 0$:

$$c_o = \frac{S_{\text{wing}}}{b} \frac{2}{1 + \lambda} = c_{\text{avg}} \frac{2}{1 + \lambda} \tag{41}$$

where $c_{\text{avg}} = S_{\text{wing}}/b$ is the average chord. Inserting this c_o into the general I expression (36) gives the root inertia

$$I_o \simeq K_I\, c_o^4\, \tau(\tau^2 + \varepsilon^2) = K_I\, c_{\text{avg}}^4 \frac{16}{(1 + \lambda)^4}\, \tau(\tau^2 + \varepsilon^2) \tag{42}$$

which can then be inserted into (29) to give an alternative tip deflection expression in terms of the average chord and the taper ratio.

$$\delta \simeq \frac{N W_{\text{cent}}}{E} \left(\frac{1}{K_I\, c_{\text{avg}}^4} \frac{(1+\lambda)^4}{16} \frac{1}{\tau(\tau^2 + \varepsilon^2)} \right) \frac{b^3}{96} \frac{1 + 2\lambda}{1 + \lambda} \tag{43}$$

$$\frac{\delta}{b} \simeq 0.018 \frac{N W_{\text{cent}}}{E \tau(\tau^2 + \varepsilon^2)} (1+\lambda)^3 (1+2\lambda) \frac{b^2}{c_{\text{avg}}^4} \tag{44}$$