18.06 Professor Edelman Quiz 1 March 2, 2018

Your PRINTED name is: ____________

Please circle your recitation:

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(2)	T 11	26-328	D. Kubrak
(3)	T 12	4-159	P.B. Alvarez
(7)	T 12	4-153	E. Belmont
(4)	T 1	4-149	P.B. Alvarez
(5)	T 2	4-149	E. Belmont
(6)	T 3	4-261	J. Wang

Grading
1
2
3
Total:

Important Instructions: We will be using Gradescope which requires your solutions appear in the boxes provided. Please place your solutions in the boxes if possible.

If you need extra pages, please write continued in the box, and on the extra pages clearly label with problem number and letter.

1 (30 pts.)

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 5 & 12 & 2 \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

(a) (15 pts.) Perform elimination on $[A\ b]$ to determine the condition that b is in the column space of A.

We can work with the augmented matrix $[A\ b]$ and row reduce as follows:

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 5 & 12 & 2 & b_3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 1 & b_1 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & -3 & -3 & b_3 - 5b_1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 3 & 1 & b_1 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & 0 & 0 & b_3 - 5b_1 - 3(b_2 - 3b_1) \end{array}\right).$$

In the first step, we subtracted 3 times the first row from the second row, and 5 times the first row from the third row. In the second step, we subtracted 3 times the second row from the third row.

Our last equation is then $0 = b_3 - 5b_1 - 3(b_2 - 3b_1)$, and simplifying gives us our condition $\boxed{4b_1 - 3b_2 + b_3 = 0}$. We notice that the first two equations are always solvable.

(b) (5 pts.) What is the rank of A?

There are two pivots in the reduced matrix, so the rank is $\boxed{2}$.

(c) (10 pts.) Find a vector in the nullspace of A^T (this space is known as the left nullspace.)

One might have noticed from the solution in (a) that $[4 \;\; -3 \;\; 1] * A = 0$ considering the column view of matrix multiply, so clearly $[4 \;\; -3 \;\; 1]^T$ is in the left nullspace.

The more time consuming way is to take the transpose of the original matrix:

$$A^T = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 8 & 12 \\ 1 & 2 & 2 \end{pmatrix}.$$

Now, we can row reduce

$$\begin{pmatrix} 1 & 3 & 5 \\ 3 & 8 & 12 \\ 1 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 5 \\ 0 & -1 & -3 \\ 0 & -1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 5 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

The third column is the only free column, so we can get an element in the nullspace by finding a vector $\mathbf{v} = \begin{pmatrix} a & b & 1 \end{pmatrix}^T$ for which the last matrix times $\mathbf{v}$ is equal to zero. Solving for a and b then give us that

$$\boxed{\mathbf{v} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}}$$

is in the nullspace of A^T.

2 (30 pts.) (6 points each part) Consider the 5x5 matrices

$$E_1 = \begin{pmatrix} 1 & . & . & . & . \\ a & 1 & . & . & . \\ b & . & 1 & . & . \\ c & . & . & 1 & . \\ d & . & . & . & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & x & 1 \end{pmatrix}, \text{and } E = E_1 E_2.$$

Here the dot (".") denotes 0.

(a) Solve $E_1 x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ for x.

The fastest way might be to recognize that $E1$ performs four trivial row operations of addiing a, b, c, d times the first row to the other rows, and $E1^{-1}$ undoes the same operations, i.e. subtracts a, b, c, d times the first row from the other rows, giving immediately $1, 1-a, 1-b, 1-c, 1-d$.

Otherwise, one can notice that E_1 is lower triangular, so we can solve this using forward substitution. Here we see $x_1 = 1$ and the other equations become

$$x_2 + a = 1$$

$$x_3 + b = 1$$

$$x_4 + c = 1$$

$$x_5 + d = 1$$

And thus the solution is $x = \begin{pmatrix} 1 \\ 1-a \\ 1-b \\ 1-c \\ 1-d \end{pmatrix}$

(b) Solve $E_1^T x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ for x.

The fastest way is to recognize $(E_1^T)^{-1} = (E_1^{-1})^T$ from above which would put the $1 - a - b - c - d$ in the first entry, and leave the other entries untouched.

More straightforwardly, E_1^T is upper triangular, so we can solve this using backwards substitution. Here we see $x_2 = x_3 = x_4 = x_5 = 1$ and the other equation becomes

$$x_1 + a + b + c + d = 1$$

And thus the solution is $x = \begin{pmatrix} 1 - a - b - c - d \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

(c) Compute E.

The fastest way is to recognize that computing E just superimposes the lower triangular entries because there is no interference between E_1 on the left and E_2 on the right with the row operations.

Otherwise one can use ones favorite matmul view on

$$\begin{pmatrix} 1 & . & . & . & . \\ a & 1 & . & . & . \\ b & . & 1 & . & . \\ c & . & . & 1 & . \\ d & . & . & . & 1 \end{pmatrix} \begin{pmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & x & 1 \end{pmatrix} = \begin{pmatrix} 1 & . & . & . & . \\ a & 1 & . & . & . \\ b & . & 1 & . & . \\ c & . & . & 1 & . \\ d & . & . & x & 1 \end{pmatrix}$$

(d) Compute E^{-1}. Check your answer.

Just as above we compute this using the following product

$$E^{-1} = E_2^{-1} E_1^{-1} = \begin{pmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & -x & 1 \end{pmatrix} \begin{pmatrix} 1 & . & . & . & . \\ -a & 1 & . & . & . \\ -b & . & 1 & . & . \\ -c & . & . & 1 & . \\ -d & . & . & . & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & . & . & . & . \\ -a & 1 & . & . & . \\ -b & . & 1 & . & . \\ -c & . & . & 1 & . \\ -d + xc & . & . & -x & 1 \end{pmatrix}$$

To check the answer we multiply this by the matrix E of part (c)

$$\begin{pmatrix} 1 & . & . & . & . \\ -a & 1 & . & . & . \\ -b & . & 1 & . & . \\ -c & . & . & 1 & . \\ -d + xc & . & . & -x & 1 \end{pmatrix} \begin{pmatrix} 1 & . & . & . & . \\ a & 1 & . & . & . \\ b & . & 1 & . & . \\ c & . & . & 1 & . \\ d & . & . & x & 1 \end{pmatrix} = \begin{pmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{pmatrix}$$

(e) Compute $(E_1)^{10} = E_1 \times E_1 \times E_1 \times E_1 \times E_1 \times E_1 \times E_1 \times E_1 \times E_1 \times E_1$.

The best solution is the operator interpretation of adding a, b, c, d times row 1, 10 times and interpreting the answer.

One can also notice the pattern and perform the induction: We do 2 multiplication steps to notice and pattern and then write the solution

$$E_1^2 = \begin{pmatrix} 1 & . & . & . & . \\ a & 1 & . & . & . \\ b & . & 1 & . & . \\ c & . & . & 1 & . \\ d & . & . & . & 1 \end{pmatrix} \begin{pmatrix} 1 & . & . & . & . \\ a & 1 & . & . & . \\ b & . & 1 & . & . \\ c & . & . & 1 & . \\ d & . & . & . & 1 \end{pmatrix} = \begin{pmatrix} 1 & . & . & . & . \\ 2a & 1 & . & . & . \\ 2b & . & 1 & . & . \\ 2c & . & . & 1 & . \\ 2d & . & . & . & 1 \end{pmatrix}$$

$$E_1^3 = \begin{pmatrix} 1 & . & . & . & . \\ 2a & 1 & . & . & . \\ 2b & . & 1 & . & . \\ 2c & . & . & 1 & . \\ 2d & . & . & . & 1 \end{pmatrix} \begin{pmatrix} 1 & . & . & . & . \\ a & 1 & . & . & . \\ b & . & 1 & . & . \\ c & . & . & 1 & . \\ d & . & . & . & 1 \end{pmatrix} = \begin{pmatrix} 1 & . & . & . & . \\ 3a & 1 & . & . & . \\ 3b & . & 1 & . & . \\ 3c & . & . & 1 & . \\ 3d & . & . & . & 1 \end{pmatrix}$$

and thus inductively we can see

$$E_1^{10} = \begin{pmatrix} 1 & . & . & . & . \\ 10a & 1 & . & . & . \\ 10b & . & 1 & . & . \\ 10c & . & . & 1 & . \\ 10d & . & . & . & 1 \end{pmatrix}$$

3 (40 pts.) Answer the following with TRUE/FALSE and explain briefly and convincingly. (1 point for the correct answer, and 3 points for a correct justification. No points for the wrong answer no matter how creative the explanation.)

(a) Is it TRUE/FALSE that the matrix $M = vv^T$ is always symmetric, when v is a vector of length n?

TRUE: (Note that v defaults to a column vector as is the convention in this class.) We can test if $M = M^T$: $(vv^T)^T = (vv^T)$ by the rules of transpose. Any matrix that is equal to its transpose is symmetric.
Alternatively one can check that $M_{ij} = v_i v_j = v_j v_i$.
Points were lost for doing only one example or only $n = 2$ or $n = 3$. Points were lost if notations were written such as v_i^T which indicated lack of understanding.

Note: in this class, "vector" means "column vector", so if you only treated the case where v was a row vector, you did not get full credit.

(b) Is it TRUE/FALSE that the matrix $M = vv^T$ might possibly have rank 0, or rank 1, but could never have rank 2?

TRUE: if we write $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ then the columns of vv^T are $v_1 v$, $v_2 v$, $\ldots$, $v_n v$. That is, they are all in the span of v. If v is the zero vector, then M has rank zero. Otherwise, there is at least some nonzero v_i, and hence at least one nonzero column of M. Then $\text{rank}(M) = \text{rank}(\text{span}(v_1 v, \ldots, v_n v)) = 1$.

Note: in this class, "vector" means "column vector", so if you only treated the case where v was a row vector, you did not get full credit.

(c) Is it TRUE/FALSE that it is possible to compute $y = ABx$ with approximately a constant multiple of n^2 operations, where A and B are general $n \times n$ matrices, and x is a general n vector?

TRUE: multiplying an $n \times n$ matrix by a column vector is $O(n^2)$. If you compute ABx as $A(Bx)$ (computing Bx first, then multiplying that on the left by A), this amounts to doing matrix-times-vector multiplication twice. Two consecutive $O(n^2)$ operations is still $O(n^2)$.

Note: multiplying two $n \times n$ matrices (using the algorithm we've been taught) is, in general, $O(n^3)$. So order matters, in terms of complexity.

(d) Is it TRUE/FALSE that in the above computation of y, if x is known to be the first column of the identity matrix, then it is generally possible to reduce the number of operations to a constant multiple of n?

FALSE: If x is the $n \times 1$ vector with a 1 in the first entry and zeros everywhere else, then Bx is just the first column of B. But then, computing $A(Bx)$ is multiplying an $n \times n$ matrix by an $n \times 1$ matrix, where we have no extra helpful information about the entries (e.g. that some of the entries are zero), and so this operation would take $O(n^2)$ time. If we were to compute in the opposite order $(AB)x$, just the AB computation would be $O(n^3)$ time. Therefore, there is no way to generally reduce the number of operations to a constant multiple of n ($O(n)$ time).

The ikj formulation of square matrix multiply has the following key lines:

```
C = zeros(n,n)
for i=1:n, k=1:n, j=1:n
    C[i,j] += A[i,k] * B[k,j]
end
```

where first $i = 1$ and $k = 1$ and then j goes from 1 to n.

Next $i = 1$ and $k = 2$ and then j goes from 1 to n, etc.

(e) Is it TRUE/FALSE that the calculation of the (1,1) entry of C is completed before the calculation of the (1,2) entry is started?

FALSE: First (i,k,j)=(1,1,1) so C_{11} gets the value of the first product $A_{11}B_{11}$, then (i,k,j)=(1,1,2) so C_{12} gets its first product $A_{11}B_{12}$. Other pieces of C_{11} of the form. Let b be a given nonzero vector of length 4. Consider the set of matrices A for which b is in the column space of A.

(f) Is it TRUE/FALSE that the ikj formulation is a ROW based view of matrix multiply?

TRUE: we are computing C one row at a time: in the first iteration of the outer loop ($i = 1$), we compute the first row of C in its entirety; once this is done, we increment i to 2 and compute the second row of C, and so on.

(g) Is it TRUE/FALSE that a 3x3 rank 2 matrix can have an RREF whose first column is all zeros?

TRUE: In order for the first column of the RREF to be 0, it has to be from the start 0 and as there are 2 non-zero columns left we can still achieve rank 2. An example of such a matrix is the following

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(h) Is it TRUE/FALSE that a 3x3 rank 2 matrix can have an RREF whose second column is all zeros?

TRUE: As above in order for the second column of the RREF to be 0, it has to be from the start 0 and as there are 2 non-zero columns left we can still achieve rank 2. An example of such a matrix is the following

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(i) Let x be a given nonzero vector of length 4. Consider the set of matrices A for which x is in the nullspace of A. Is it TRUE/FALSE that this set of matrices form a vector space?

TRUE: We accepted true as the correct answer, with the unstated assumption that the matrices all had the same dimension such as 4×4. With this assumption, 0 is in the space, and if A and B are in the space, all linear combinations, including sums and scalar multiples are in the space. Full credit required clarity about sums and scalar multiples.

False was accepted, but only if there was a fully correct explanation, that matrices with different dimensions can not be added, and thus can not form a vector space.

(j) Let b be a given nonzero vector of length 4. Consider the set of matrices A for which b is in the column space of A. Is it TRUE/FALSE that this set of matrices form a vector space?

FALSE: The easiest explanation is to point out that the zero matrix is not in this space.