18.06 Professor Edelman Quiz 3 May 4, 2018

Your PRINTED name is: ____________

Please circle your recitation:

(1)	T 10	26-328	D. Kubrak
(2)	T 11	26-328	D. Kubrak
(3)	T 12	4-159	P.B. Alvarez
(7)	T 12	4-153	E. Belmont
(4)	T 1	4-149	P.B. Alvarez
(5)	T 2	4-149	E. Belmont
(6)	T 3	4-261	J. Wang

Note: We are not planning to use gradescope for this exam.

Grading	
1	
2	
3	
Total:	

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1 (30 pts.) Consider the matrices,

$$A(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + t \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

a. (5 pts) Is it possible to find a vector v and a scalar λ that does not depend on t that serves as an eigenvector/eigenvalue for $A(t)$ for all t?

SOLUTION:

$A(t)$ has eigenvalue $\lambda_1 = 1$ with eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for all t. One way to see this is that v_1 is in the null space of $t \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ (so $A(t)v_1$ does not depend on t), and it also happens to be an eigenvector of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Alternatively, you could compute and factor the characteristic polynomial as we do in b.

b. (5 pts.) Find both an eigenvector and an eigenvalue of $A(t)$ that does depend on t.

SOLUTION:

The characteristic polynomial of $A(t) = \begin{bmatrix} t & 1-t \\ 1 & 0 \end{bmatrix}$ is

$$\begin{aligned} \det(A(t) - \lambda I) &= (t - \lambda)(-\lambda) - (1 - t) \\ &= \lambda^2 - \lambda t + (t - 1) \\ &= (\lambda - 1)(\lambda - (t - 1)). \end{aligned}$$

(To help you factor this, you could use the fact from (a) that 1 is a root of the characteristic polynomial.) So $\lambda_2 = t - 1$, and

$$A(t) - \lambda_2 I = A(t) - (t-1)I = \begin{bmatrix} 1 & 1-t \\ 1 & 1-t \end{bmatrix}$$

has null space spanned by $v_2 = \begin{bmatrix} t-1 \\ 1 \end{bmatrix}$, so that is the corresponding eigenvector.

c. (5 pts.) For which t, if any, is the matrix $A(t)$ not diagonalizable. Explain briefly.

SOLUTION:

If $\lambda_1 \neq \lambda_2$, then $A(t)$ is diagonalizable. So the only case in which non-diagonalizability is possible is when $1 = t - 1$, i.e., $t = 2$. In this case, the characteristic polynomial is $(\lambda - 1)^2$, and $A(2)$ is diagonalizable if and only if $\dim N(A(2) - I) = 2$. But $A(2) - I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ which has rank 1 and null space of dimension 1. So it is not diagonalizable at $t = 2$.

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d. (5 pts.) Consider the sequence $x_0 = 0$, $x_1 = 1$, $x_{k+2} = t * x_{k+1} + (1-t) * x_k$. You can assume $0 < t < 2$. Why does x_k converge to a finite number as $k \to \infty$? Explain briefly.

SOLUTION: Briefly, this happens because $A(t)$ has eigenvalues 1 and $t-1$ and since $0 < t < 2$ we have $|t-1| < 1$.

In more detail, the recursion on x_k's can be rewritten as

$$\begin{bmatrix} x_{k+2} \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} t & 1-t \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix}$$

so we encounter our matrix $A(t)$ and get a formula

$$\begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} = A(t) \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix} = A(t)^2 \begin{bmatrix} x_{k-1} \\ x_{k-2} \end{bmatrix} = \ldots = A(t)^k \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = A(t)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Recall that $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} t-1 \\ 1 \end{bmatrix}$ are eigenvectors of $A(t)$ with eigenvalues 1 and $t-1$. By part c if $t \neq 2$ the matrix $A(t)$ is diagonalizable and v_1, v_2 form an eigenbasis. So we can express our vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as $c_1(t)v_1 + c_2(t)v_2$ for some scalars $c_1, c_2 \in \mathbb{R}$ that depend on t. Now we have

$$A(t)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A(t)^k(c_1v_1 + c_2v_2) = c_1(A(t)^kv_1) + c_2(A(t)^kv_2) = c_1v_1 + c_2(t-1)^kv_2$$

Since $|t-1| < 1$ we have $\lim_{k\to\infty} c_1v_1 + c_2(t-1)^kv_2 = c_1v_1$, in particular the limit is a finite vector. Also we get that

$$\lim_{k\to\infty} \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} = c_1v_1 = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix}$$

Remark: Note that $A(t)$ is not necessarily Markov since $1-t$ can be negative (if $t > 1$). To use that was a pretty common mistake on the exam.

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e. (10 pts.) (Recommended to do this after completing all other work on the exam.) Calculate the limit of x_k from part d as k goes to infinity. (Hint: Consider the vector $\begin{pmatrix} x_{k+1} \\ x_k \end{pmatrix}$.) (Check: If $t = 1/2$ the limit is $2/3$.)

SOLUTION: By what we proved in part d)

$$\lim_{k\to\infty} \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} = c_1 v_1 = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix}$$

where $c_1 v_1 + c_2 v_2$ is the decomposition of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in the eigenbasis v_1, v_2. So to find the limit it is enough to find such decomposition. For this note that

$$v_2 - v_1 = \begin{bmatrix} t-1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} t-2 \\ 0 \end{bmatrix} = (t-2)\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ so } \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{-1}{t-2} v_1 + \frac{1}{t-2} v_2$$

We get that $c_1 = \dfrac{1}{2-t}$ and so $\lim_{k\to\infty} x_k = \dfrac{1}{2-t}$. In particular for $t = 1/2$ we get $2/3$.

Remark: Another way to do this problem was to diagonilize matrix $A(t)$, since v_1, v_2 are eigenvectors with eigenvalues $1, t-1$ we have

$$A(t) = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & t-1 \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}^{-1} = \begin{bmatrix} 1 & t-1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & t-1 \end{bmatrix} \frac{1}{2-t} \begin{bmatrix} 1 & 1-t \\ -1 & 1 \end{bmatrix}$$

and so

$$A(t)^k = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (t-1)^k \end{bmatrix} \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}^{-1} = \begin{bmatrix} 1 & t-1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (t-1)^k \end{bmatrix} \frac{1}{2-t} \begin{bmatrix} 1 & 1-t \\ -1 & 1 \end{bmatrix} =$$

$$= \frac{1}{2-t} \begin{bmatrix} 1 & (t-1)^{k+1} \\ 1 & (t-1)^k \end{bmatrix} \begin{bmatrix} 1 & 1-t \\ -1 & 1 \end{bmatrix}$$

When k goes to infinity $(t-1)^k$ dies we get

$$\lim_{k\to\infty} A(t)^k = \frac{1}{2-t}\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1-t\\ -1 & 1\end{bmatrix} = \frac{1}{2-t}\begin{bmatrix}1 & 1-t\\ 1 & 1-t\end{bmatrix}$$

Finally

$$\begin{bmatrix}\lim_{k\to\infty} x_{k+1}\\ \lim_{k\to\infty} x_k\end{bmatrix} = \lim_{k\to\infty} A(t)^k \begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}1/(2-t)\\ 1/(2-t)\end{bmatrix}$$

So we get

$$\lim_{k\to\infty} x_k = \frac{1}{2-t}.$$

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2 (30 pts.) In all cases find a two by two matrix which has the given eigenvalues and the given singular values or explain why it is impossible. Do not use $A^T A$ or AA^T in any of your explanations.

a. (5 pts) $\lambda = 0, 1$**,** $\sigma = 1, 1$

b. (5 pts) $\lambda = 0, 1$ $\sigma = 0, \sqrt{2}$

c. (5 pts) $\lambda = 0, 0$ $\sigma = 0, 2018$

d. (5 pts) $\lambda = i, -i$ $\sigma = 1, 1$

e. (5 pts) $\lambda = 4, 4$ $\sigma = 3, 5$

f. (5 pts.) $\lambda = -1, 1$ $\sigma = \sqrt{(3 \pm \sqrt{5})/2}$ **(You can trust that** $\sigma_1\sigma_2 = 1$ **and** $\sigma_1^2 + \sigma_2^2 = 3$**)**

SOLUTION:

Throughout these solutions, we will use the following facts. If λ_1 and λ_2 are the eigenvalues and σ_1 and σ_2 are the singular values of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

1. $\sigma_1 \cdot \sigma_2 = |\det(A)| = |\lambda_1 \cdot \lambda_2|$.

2. $\sigma_1^2 + \sigma_2^2 = a^2 + b^2 + c^2 + d^2$.

3. An upper triangular matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ has eigenvalues a and d.

Now, we have the following.

(a) Since one of the eigenvalues is 0, the matrix A cannot have full rank. But both singular values are nonzero, so the matrix has rank 2. This is a contradiction, so A does not exist.

(b) The matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

satisfies these conditions. It is upper triangular, so by fact 3, the eigenvalues are 0 and 1. The singular values have to satisfy $\sigma_1 \cdot \sigma_2 = 0$ and $\sigma_1^2 + \sigma_2^2 = (-1)^2 + 1^2 = 2$ by facts 1 and 2. The solution to this pair of equations is unique (up to reordering σ_1 and σ_2), and so the singular values are $\sqrt{2}$ and 0.

(c) We claim that the matrix

$$A = \begin{bmatrix} 0 & 2018 \\ 0 & 0 \end{bmatrix}$$

satisfies these conditions. It is upper triangular, so by fact 3, the eigenvalues are 0 and 0. By facts 1 and 2, we have that $\sigma_1 \cdot \sigma_2 = 0$ and $\sigma_1^2 + \sigma_2^2 = 2018^2$, and so the singular values are 0 and 2018.

(d) Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

This is a rotation counterclockwise by 90 degrees. We can compute its eigenvalues by calculating $\det(A - \lambda I) = \lambda^2 + 1 = 0$ to get $\lambda = \pm i$. We can see geometrically that a rotation has singular values 1 and 1 since it rotates a unit circle in $\mathbb{R}^2$ onto the unit circle again. Alternatively, we can solve for $\sigma_1 \cdot \sigma_2 = 1$ and $\sigma_1^2 + \sigma_2^2 = (-1)^2 + 1^2 = 2$ to get that the singular values are 1 and 1.

(e) Since $\lambda_1 \cdot \lambda_2 = 16$ and $\sigma_1 \cdot \sigma_2 = 15 \neq |16|$, by fact 1, a matrix A with these eigenvalues and singular values does not exist.

(f) Let

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The matrix is upper triangular and so by fact 3 has eigenvalues -1 and 1. By facts 1 and 2, the singular values satisfy $\sigma_1 \cdot \sigma_2 = 1$ and $\sigma_1^2 + \sigma_2^2 = (-1)^2 + 1^2 + 1^2 = 3$, and so must be the $\sigma_i = \sqrt{(3 \pm \sqrt{5})/2}$.

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3 (40 pts.) Are the following matrices necessarily positive definite? Explain why or why not?

a. (5 pts) $A = Q\Lambda Q^T$ **where** Q **is some 4x4 orthogonal matrix and** Λ **is diagonal with** $(1, 2, 3, 4)$ **on the diagonal.**

b. (10 pts) $A = Q_1\Lambda Q_1^T + Q_2\Lambda Q_2^T$**, where** Q_1 **and** Q_2 **are some 4x4 orthogonal matrices and** Λ **is diagonal with** $(1, 2, 3, 4)$ **on the diagonal.**

c. (5 pts) $A = X\Lambda X^T$ **for some matrix** X **and** Λ **is as above? (Hint: Be careful.)**

d. (5 pts.) P **the projection matrix onto** $(1, 2, 3, 4)$**.**

e. (15 pts.) A **is the** n **by** n **tridiagonal matrix with** 2 **for each diagonal entry, and** 1 **for each superdiagonal and subdiagonal entry.** $n = 1, 2, 3, \ldots$**. (Hint: Probably the easiest argument involves computing the determinant of** $T(n)$ **for** $n = 1, 2, 3, \ldots$**.)**

SOLUTION:

(a) First this matrix is clearly symmetric as $A^T = A$. Since Q is orthogonal $Q^T = Q^{-1}$ and so this gives a diagonalization, so A has eigenvalues $1, 2, 3, 4 > 0$ and is thus positive definite.

(b) From part a) we know that each of $A_i = Q_i\Lambda Q_i^T$ are positive definite, so for $x \neq 0$ $x^T A_i x > 0$. So we get $x^T Ax = x^T A_1 x + x^T A_2 x > 0$ and it follows that A is positive definite as A is clearly symmetric as it is the sum of symmetric matrices.

(c) We have $A^T = A$, so A is symmetric. Also $v^T Av = (X^T v)^T \Lambda(X^T v) \geq 0$ as Λ positive definite and the inequality is strict as long as $X^T v \neq 0$. So we get that A is not positive definite as long as X^T is not full column rank.

(d) This is a projection matrix to a 1 dimensional space, so P has rank 1. It thus has a non-trivial nullspace and so has a 0 eigenvalue. So not all eigenvalues are positive and thus is not positive definite.

(e)

$$A_n = \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix}$$

Lets denote by $T(n)$ the determinant of the nxn matrix A_n. By expanding the determinant along the first column, we get the formula

$$T(n) = 2T(n-1) - det(\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}) = 2T(n-1) - T(n-2)$$

Were we expand the second determinant along the first row. Also $T(1) = 2$ and $T(2) = 3$, so we can check $T(n) = n+1 > 0$. So we have A_n is symmetric and all top left corner determinants are positive so it is positive definite.

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