2.003 Fall 1999 Homework Assignment 10

1. **Dynamic braking in an electric motor.** (24 points, 12 points for part a, 4 points for part b, 8 points for part c)The motor constants supplied by the manufacturer are in a variety of inconsistent units. It is helpful to transform them into a single consistent set of units. In SI units:

$$\begin{aligned}
K_T &= 5.03 \text{ oz-in/Amp} &&= 5.03(0.278)(0.0254) &&= 0.0355 \text{ N-m/Amp} \\
K_e &= 3.72 \text{ Volts/Krpm} &&= 3.72(60/2\pi)/1000 &&= 0.0355 \text{ Volts/rad/sec} \\
B_m &= 0.59 \text{ oz-in/Krpm} &&= 0.59(0.278)(0.0254)(60/2\pi)/1000 &&= 3.98\text{e-5 N-m/rad/sec} \\
I_m &= 0.0028 \text{ oz-in-sec}^2 &&= 0.0028(0.278)(0.0254) &&= 1.977\text{e-5 N-m-sec}^2
\end{aligned}$$

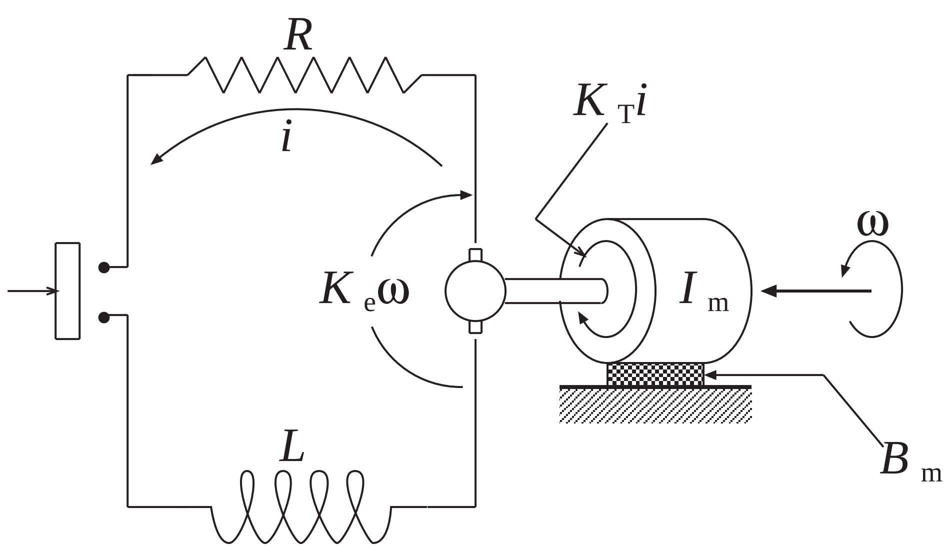

Figure 1: Electric Motor

At $t = 0$ the shaft is rotating at 1000 rpm [1000 $(2\pi/60)$ = 104.7 rad/sec]. On the mechanical side the rotating inertia is decelerated by the friction torque $B_m\omega$ and the motor torque $K_T i$

$$I_m \frac{d\omega}{dt} + B_m\omega = -K_T i$$

and on the electrical side, the back-emf generated by the motion of the armature causes a current i to flow through the resistance R (R is the sum of the armature resistance R_a and the terminal resistance R_t: $R = R_a + R_t = 1.12 + 1.40 = 2.52$ ohms) and the inductance L when the circuit is closed

$$L\frac{di}{dt} + Ri = K_e\omega$$

All we know about the inductance L is that it is less than 100 micro-henries. This suggests that the inductance may not be very important. A way to confirm this is to consider the decay time-constants defined by the preceding equations. On the mechanical side

$$\tau_m = \frac{I_m}{B_m} = \frac{1.977}{3.98} = 0.497 secs$$

while on the electrical side

$$\tau_e = \frac{L}{R} < \frac{100\text{e-}6}{2.52} = 0.0000397 secs$$

The decay time on the electrical side is so short in comparison with that on the mechanical side, that we can neglect the inductance and assume that the current i is completely determined by the back-emf and the resistance, according to the following approximate relation

$$i = \frac{K_e \omega}{R}$$

(a) The coast-down of the rotor from an initial angular velocity of 104.7 rad/sec for the open-circuit case is quite different from that for the closed-circuit case.

(i) For the open-circuit case there is no current flow ($i = 0$) and the coast-down is governed by the mechanical equation

$$I_m \frac{d\omega}{dt} + B_m \omega = -K_T i = 0$$

The solution is

$$\omega = 104.7 \exp(-\frac{B_m}{I_m} t) = 104.7 \exp(\frac{t}{\tau_o} t)$$

where $\tau_o = I_m/B_m = 0.497$ sec. This is a simple exponential decay which is 98% complete in $4\tau_o = 1.988$ seconds.

(ii) For the closed-circuit case, there is a current $i(t)$ flowing in the electrical circuit. When the current $i(t)$ is eliminated between the mechanical equation and the approximate electrical relation, the governing equation becomes

$$I_m \frac{d\omega}{dt} + B_m \omega = -K_T i = -\frac{K_T K_e}{R} \omega$$

or

$$I_m \frac{d\omega}{dt} + (B_m + \frac{K_T K_e}{R}) \omega = 0$$

The solution here is also a simple exponential decay

$$\omega = 104.7 \exp(\frac{t}{\tau_c} t)$$

but the decay time-constant now is

$$\tau_c = \frac{I_m}{B_m + \frac{K_T K_e}{R}} = \frac{1.977\text{e-}5}{3.98\text{e-}5 + \frac{(0.0355)^2}{2.52}} = 0.0366 \text{ seconds}$$

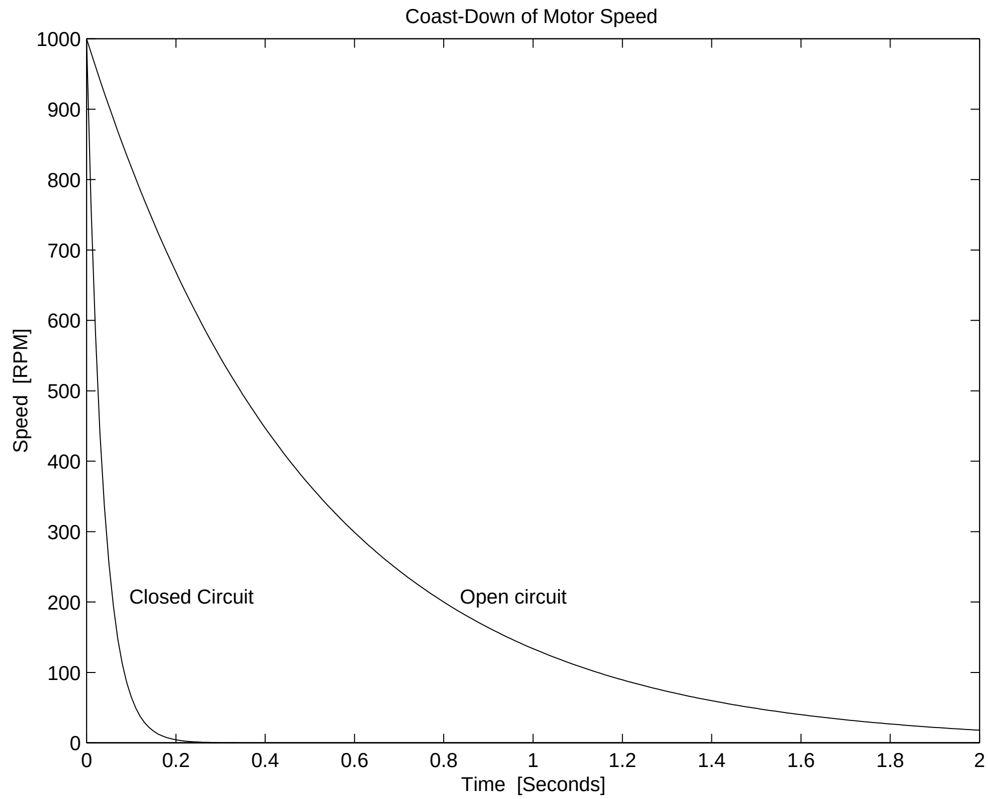

Figure 2: Dynamic Braking of Electric Motor

and the decay is 98% complete in 0.1464 seconds. The dynamic braking reduces the decay time by a factor of 13.57!

(b) The decay curves shown in Fig.2 were obtained by running the following MATLAB script:

```
% DynBrak.m  Script plots coast-down of motor for open-circuit
% case and closed-circuit case.
TauOp = 0.497;
TauCl = 0.0366;
t = 0 : 0.01 : 2.0;
plot( t, 1000*exp(-t./TauOp)), title('Coast-Down of Motor Speed'),
xlabel('Time  [Seconds]'),
ylabel('Speed  [RPM]'), pause
hold on
plot( t, 1000*exp(-t./TauCl)),
gtext('Open circuit')
gtext('Closed Circuit'), pause
hold off
clf
```

(c) The effective (mechanical) damping coefficient due to closing the electric circuit is

$$B_e = \frac{K_T K_e}{R} = \frac{(0.0355)^2}{2.52} = 5.00\text{e-4 N-m/rad/sec}$$

This is the extra term added to B_m in the differential equation for the decay of ω in the closed circuit case. Another way to arrive at B_e is to imagine that the only element on the electrical side is the resistance R which has the constitutive equation $e = iR$. Consider interrogating this element from the mechanical side. Apply a rotational velocity ω to the motor shaft. This generates a back-emf of $e = K_e\omega$ in the electrical side, which produces a current $i = e/R = K_e\omega/R$ through the resistor. This current on the electrical side generates a torque on the mechanical side of magnitude $T = K_T i = (K_T K_e/R)\omega$. Thus, as seen from the mechanical side, the presence of a resistor R on the electrical side causes a rotational velocity ω to be resisted by a torque $T = (K_T K_e/R)\omega$. A mechanical damper with damping coefficient $B_e = (K_T K_e/R)$ would produce the same resisting torque to an applied angular velocity ω.

2. **Equivalent moment of inertia of a capacitor.** (26 points, 10 points each for parts a and c, 6 points for part b) A 30,000 μF capacitor C is connected to the terminals of the preceding DC permanent magnet motor and a suddenly applied constant torque of $T_a = 4$ N-m is applied to the shaft of the motor. In this mode of operation the device acts more like a generator than a motor.

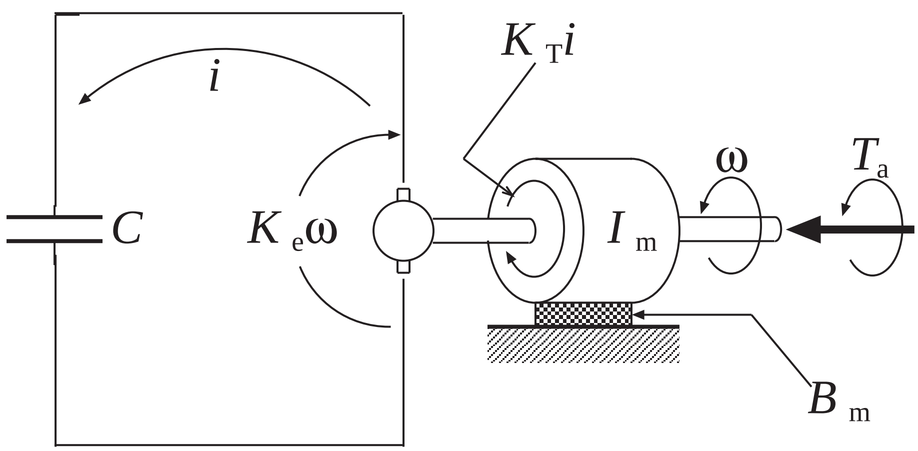

Figure 3: Charging a Capacitor with an Electric Motor

(a) In the simplest model, we neglect the armature resistance and inductance and the terminal resistance, and consider that the only element on the electrical side is the capacitor C. On the mechanical side, we include the rotor inertia I_m and the linear friction coefficient B_m. The mechanical equation of motion is

$$T_a = I_m \frac{d\omega}{dt} + B_m\omega + K_T i$$

and on the electrical side we have

$$K_e\omega = \frac{q}{C} \qquad \text{and} \qquad \frac{dq}{dt} = i$$

from which we obtain

$$i = K_e C \frac{d\omega}{dt}$$

When the current i from the electrical side is inserted in the the mechanical equation the result is a first-order differential equation for the shaft speed ω

$$(I_m + K_T C K_e)\frac{d\omega}{dt} + B_m\omega = T_a$$

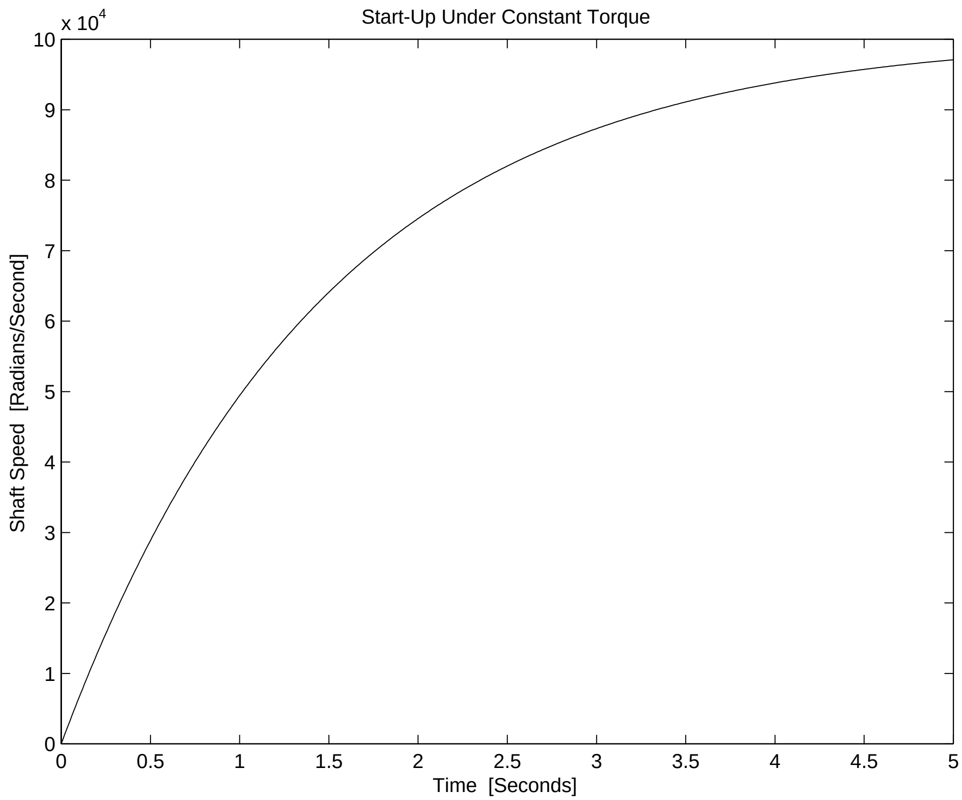

Figure 4: Motor Speed When Charging a Capacitor

Without solving the equation, we can identify the decay time-constant τ

$$\tau = \frac{I_m}{B_m} + \frac{K_T C K_e}{B_m} = \frac{1.977\text{e-}5}{3.98\text{e-}5} + \frac{(0.0355)^2(0.03)}{3.98\text{e-}5} = 0.497 + 0.950 = 1.447 \text{ seconds}$$

and the steady-state speed ω_{ss}

$$\omega_{ss} = \frac{T_a}{B_m} = \frac{4}{3.98\text{e-}5} = 100500 \text{ rad/sec}$$

(b) The solution $\omega(t)$ of the first-order differential equation which starts with $\omega = 0$ at $t = 0$, when the torque T_a is suddenly applied, is

$$\omega(t) = \omega_{ss}(1 - \exp\{-t/\tau\})$$

The time history of shaft speed for the first five seconds is displayed in Fig.4.

(c) From the first-order differential equation in (b), or from the expression for the decay time-constant τ, we see that the term $K_T C K_e$ plays the same role as the rotor moment of inertia I_m. The rotational equivalent of a capacitor C is an inertia $I_e = K_T C K_e$.
Another way to arrive at I_e is to consider that the only element on the electrical side is the capacitance C which has the constitutive equation $e = q/C$. Consider interrogating this element from the mechanical side. Apply a rotational acceleration $d\omega/dt$ to the motor shaft. This generates a rate of change of back-emf $de/dt = K_e d\omega/dt$ in the electrical side, which begins to charge the capacitor. The current $i = dq/dt = C de/dt = C K_e d\omega/dt$, on the electrical side, generates a torque on the mechanical side of magnitude $T = K_T i = (K_T C K_e) d\omega/dt$. Thus, as seen from the mechanical side, the presence of a capacitor C on the electrical side causes a rotational acceleration $d\omega/dt$ to be resisted by a torque $T = (K_T C K_e) d\omega/dt$. A mechanical inertia with moment of inertia $I_e = (K_T C K_e)$ would produce the same resisting torque to an applied angular acceleration $d\omega/dt$.

3. **Bicycle lamp powered by an AC generator.** (24 points, 10 points each for parts a and b, 4 points for part c) An AC generator generates a voltage $e(t) = A\omega \sin \omega t$ in a circuit containing an inductance of 140 millihenries and a resistance of 10 Ohms within the generator,and an output device, an electric light bulb. The resistance R_o of the light bulb is determined by a test in which a DC source of 60 volts caused the light to consume 40 Watts. Since $e_o = R_o i_o$ the power dissipated P_{diss} is

$$P_{diss} = e_o i_o = \frac{e_o^2}{R_o}$$

and thus

$$R_o = \frac{e_o^2}{P_{diss}} = \frac{60^2}{40} = 90 \text{ Ohms}$$

The connection between the frequency ω of the generator voltage and the velocity v of the bicycle is easily obtained, once it is realized that the relative velocity of the rim of the bicycle wheel past a point fixed on the frame of the bicycle is exactly the same as the velocity $v = R_w \Omega$ of the bicycle frame with respect to the road (R_w is the radius of the bicycle wheel and Ω is the angular velocity of the bicycle wheel. When the wheel of radius r on the generator shaft rolls without slip on the bicycle wheel rim its angular velocity is $\omega = v/r$. When v is in miles per hour and $r = 0.25$ inches, then

$$\omega = \frac{(5280)(12)}{(0.25)(3600)} v = 70.4 v \text{ radians/second}$$

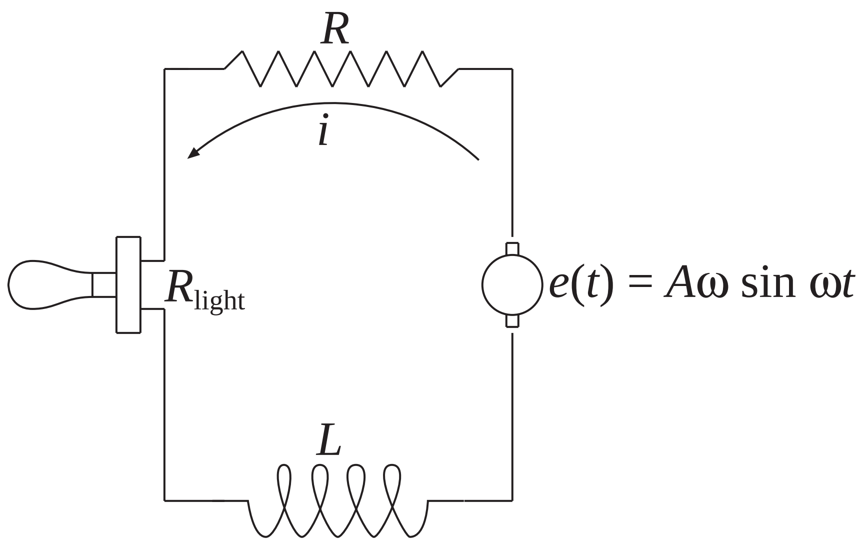

Figure 5: AC Generator Circuit

When a current i flows in the circuit, the voltage drop across the resistance R is Ri, the drop across the inductance L is Ldi/dt, and the drop across the light bulb is $R_o i$. Kirchhoff's voltage law then requires

$$L\frac{di}{dt} + (R + R_o)i = A\omega \sin \omega t = A\omega \text{Im}\{\exp(i\omega t)\}$$

This is a first-order differential equation for the current i in the circuit.

(a) The output voltage e_o across the light bulb is $R_o i$, so a first-order equationm for $e_o(t)$ can be obtained by multiplying every term in the preceding equation by R_o, and writing e_o for $R_o i$ to get

$$L\frac{de_o}{dt} + (R + R_o)e_o = R_o A\omega \sin \omega t = R_o A\omega \text{Im}\{\exp(i\omega t)\}$$

(b) At a fixed speed ω, the steady-state solution to the previous differential equation may be taken in the form

$$e_o = \text{Im}\{A_o \exp(i\omega t)\}$$

When this is substituted in the equation, the exponential terms cancel out, leaving a simple algebraic equation for the complex response amplitude A_o, whose solution, in terms of the angular speed ω of the generator shaft, is

$$A_o = \frac{R_o A\omega}{(R + R_o) + iL\omega} = \frac{(90)(0.12)\omega}{(10 + 90) + i(0.14)\omega} = \frac{10.8\omega}{100 + i(0.14)\omega}$$

In terms of the bicycle speed v in miles per hour (using $\omega = 70.4v$) the complex amplitude of the voltage across the light bulb is

$$A_o = \frac{760v}{100 + i9.86v}$$

The magnitude of A_o is

$$|A_o| = \frac{760v}{\sqrt{(100)^2 + (9.86v)^2}}$$

and the phase is

$$\phi = \arctan -\frac{9.86v}{100}$$

(c) The light bulb voltage is an alternating voltage of the form

$$e_o(t) = |A_o| \sin(\omega t + \phi)$$

The power absorbed by the light bulb is

$$P_o = \frac{e_o^2}{R_o} = \frac{|A_o|^2}{R_o} \sin^2(\omega t + \phi) = \frac{|A_o|^2}{2R_o}[1 - \cos(2\omega t + 2\phi)]$$

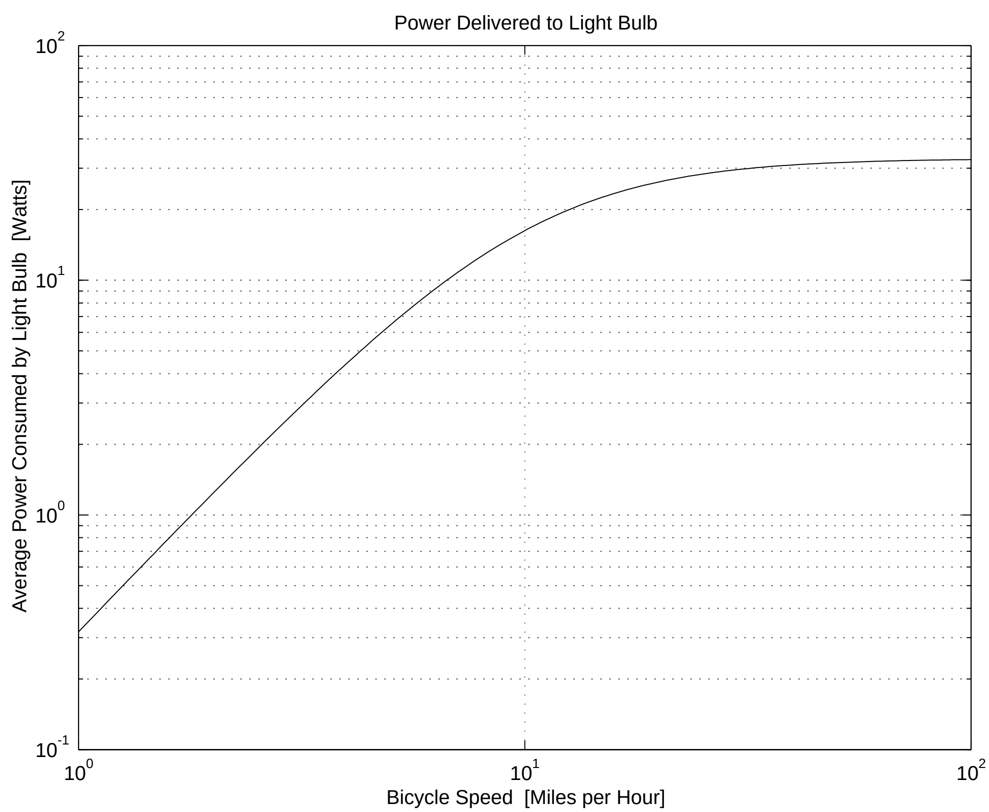

Figure 6: Power Transmitted to Light Bulb from AC Generator

Note that the power absorbed is non-negative. It fluctuates about the average value

$$P_{o,avg} = \frac{|A_o|^2}{2R_o}$$

at *twice* the frequency of the alternating voltage. The average power absorbed by the light bulb varies with the speed v of the bicycle according to the formula

$$P_{o,avg} = \frac{(760v)^2}{2(90)[(100)^2 + (9.86v)^2]} \text{ Watts}$$

This relation is displayed in Fig.6 for the speed range from 1 mph to 100 mph. The average power delivered to the light bulb increases from 1.2 Watts at 2 mph to 26 Watts at 20 mph which probably translates into pretty feeble illumination for the cyclist.

4. **Use of resonance to increase light output.** (26 points, 10 points each for parts a and b, 6 points for part c) Because of the poor performance of the design considered in Problem 3, O'Veroptimistic proposes an improved design containg a capacitance $C = 5\mu$F.

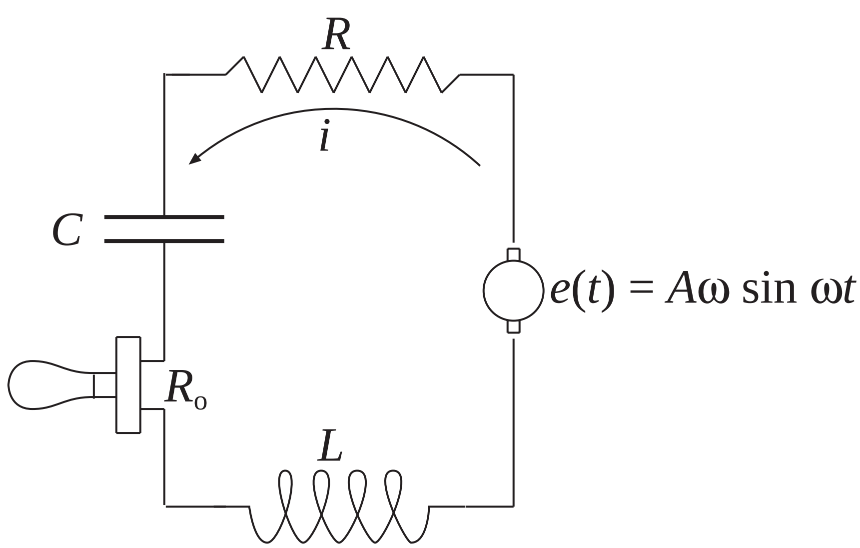

Figure 7: AC Generator Circuit with Added Capacitance

The capacitor C introduces the additional constitutive equation $e_C = q/C$ and the compatibility relation $i = dq/dt$. Kirchhoff's voltage law now becomes

$$L\frac{di}{dt} + (R + R_o)i + \frac{q}{C} = e(t) = A\omega \sin \omega t$$

(a) Taking q and i as state variables the state equations can be cast in the standard matrix form

$$\frac{d}{dt}\begin{Bmatrix} q \\ i \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R+R_o}{L} \end{bmatrix}\begin{Bmatrix} q \\ i \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{1}{L} \end{Bmatrix} A\omega \sin \omega t$$

The desired output quantities are:

(i) the capacitor voltage $e_C = q/C$, and

(ii) the light-bulb voltage $e_o = R_o i$.

In matrix form, the output equations are

$$\left\{ \begin{array}{c} e_C \\ \\ e_o \end{array} \right\} = \left[\begin{array}{cc} \frac{1}{C} & 0 \\ \\ 0 & R_o \end{array} \right] \left\{ \begin{array}{c} q \\ \\ i \end{array} \right\}$$

(b) At a fixed speed ω, the steady-state solution to the previous differential equation may be taken in the form

$$\left\{ \begin{array}{c} q \\ \\ i \end{array} \right\} e_o = \mathrm{Im}\{ \left\{ \begin{array}{c} A_q \\ \\ A_i \end{array} \right\} \exp(i\omega t)\}$$

When this is substituted in the equation, the exponential terms cancel out, leaving a pair of algebraic equations for the complex response amplitudes A_q and A_i, whose solution, in terms of the angular speed ω of the generator shaft, is

$$\left\{ \begin{array}{c} A_q \\ \\ A_i \end{array} \right\} = \left\{ \begin{array}{c} C \\ \\ i\omega C \end{array} \right\} \frac{A\omega}{1 + i\omega(R + R_o) - \omega^2 LC}$$

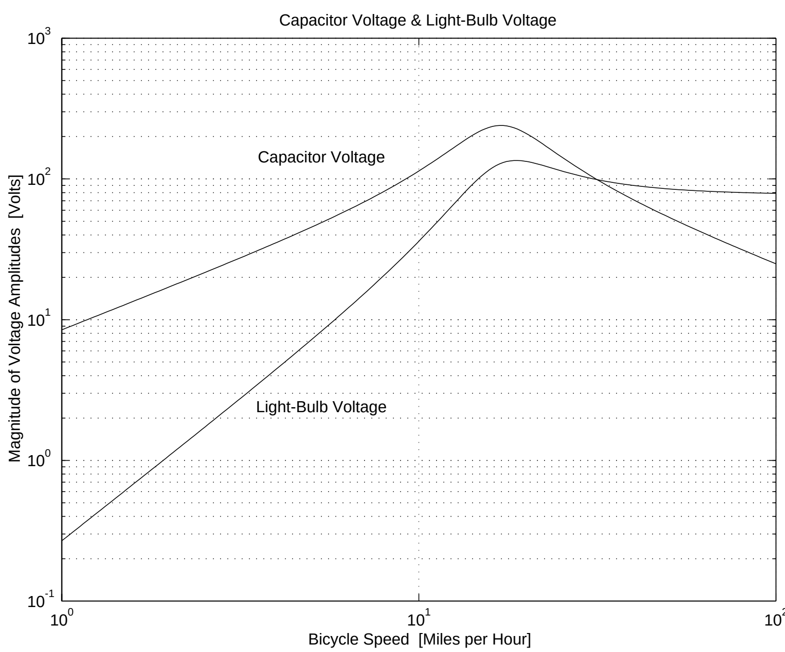

Figure 8: Voltage Amplitudes Across Capacitor & Light Bulb

The amplitudes of the output quantities then follow from

$$\left\{ \begin{array}{c} A_C \\ \\ A_o \end{array} \right\} = \left[\begin{array}{cc} \frac{1}{C} & 0 \\ \\ 0 & R_o \end{array} \right] \left\{ \begin{array}{c} A_q \\ \\ A_i \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ \\ i\omega R_o C \end{array} \right\} \frac{A\omega}{1 + i\omega(R + R_o)C - \omega^2 LC}$$

The magnitudes of these voltages are plotted as functions of bicycle speed v using the relation $\omega = 70.4v$ in Fig.8.

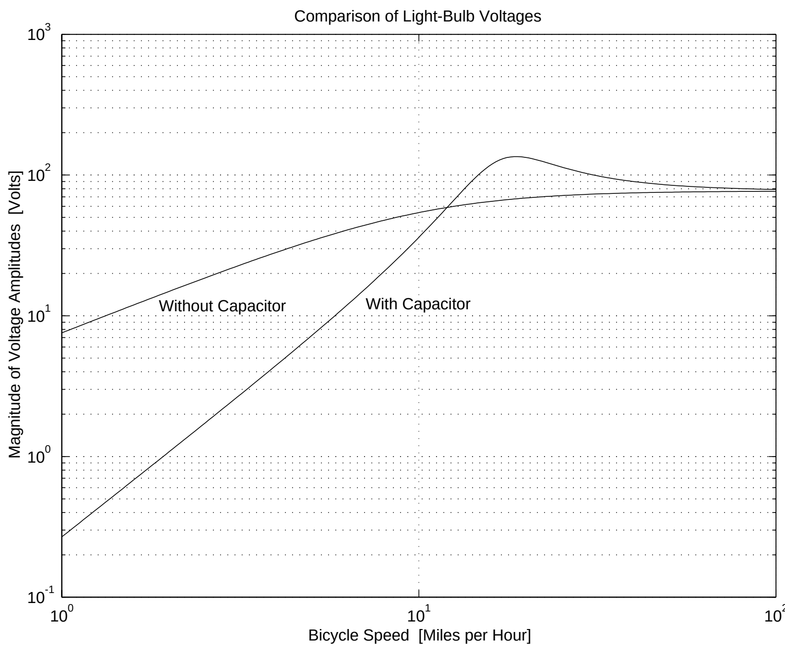

Figure 9: Light-Bulb Voltage Amplitudes with & without Capacitor in Circuit

(c) To judge whether this design represents an improvement over the design in Problem 3, compare the voltage amplitude across the light bulb for the two cases. When there is no capacitor in the circuit the complex amplitude of the voltage across the light bulb is

$$A_o(\mathrm{no}C) = \frac{R_o A\omega}{(R + R_o) + iL\omega}$$

With the capacitor C the complex amplitude of the light-bulb voltage is

$$A_o(\mathrm{with}C) = \frac{(i\omega R_o C)(A\omega)}{1 + i\omega(R + R_o)C - \omega^2 LC}$$

To see that the second of these approaches the first, divide numerator and denominator by $i\omega C$ and consider the limit as $\omega C \to \infty$. The magnitudes of light-bulb voltage amplitudes for these two cases are compared in Fig.9.

For speeds above 13 mph, inclusion of the capacitor increases the voltage across the light bulb. At 20 mph the voltage is doubled (and the wattage consumed *quadrupled*!). So Mr O'Veroptimistic was right, at least partly. The bad news is that at speeds less than 13 mph the inclusion of the capacitor decreases the voltage across the light bulb. At 8 mph the voltage is halved (and the wattage consumed reduced by a factor of 4). The results displayed in Fig.9 suggest that

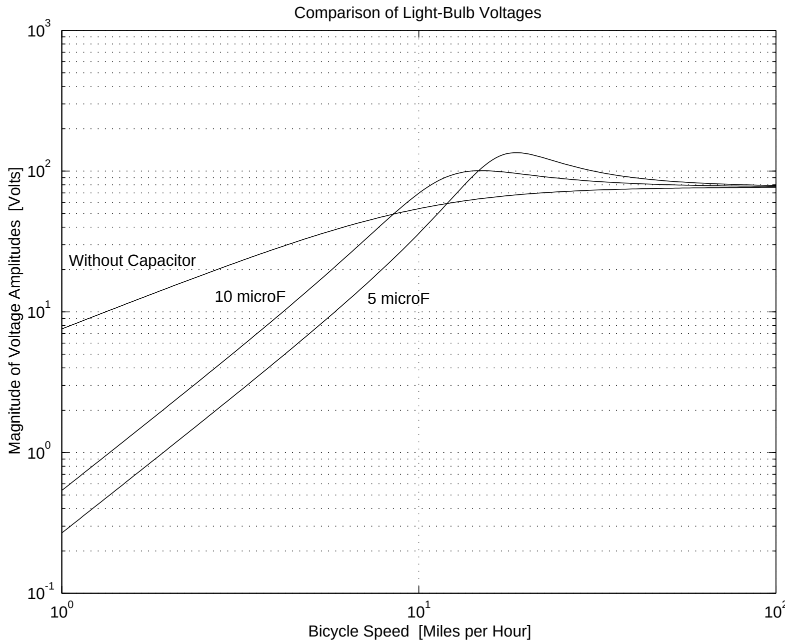

Figure 10: Light-Bulb Voltage Amplitudes with Capacitors in Circuit

the region of increased performance might be shifted to a more useful speed range by trying different values of the capacitance C. This is a typical tuning procedure required in design. The light-bulb voltage obtained with a 10 μF capacitor are compared with the previous result for a 5μF capacitor in Fig.10. With 10μF the inclusion of the capacitor improves the light output for speeds greater than 8 mph, but the increase in output is not as great as with the inclusion of a 5 μF capacitor. The optimum design requires a trade-off between the speed range of improved illumination and the actual amount of increase in illumination.