2.003 Fall 1999 Solution of Homework Assignment 9

1. (27 points, 5 points each for parts a, c and d, 6 points each for parts b and e.)Weight of engine block $W = 200$ pounds, stiffness $k = 18,000$ pounds/inch, damping coefficient $b = 2$ pounds /inch/second. Input force is $f(t) = f_a \sin \Omega t$ with $f_a = 2$ pounds.

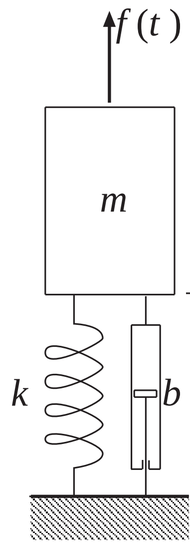

Figure 1: Engine Block Subjected to Sinusoidally Varying Force

Let $y(t)$ be displacement of engine block from equilibrium position. Geometric compatibility requires that the block velocity $v(t)$ satisfy $v = dy/dt$. Restoring force is $f_k = ky$, damping force is $f_{fric} = bv$, input force is $f(t) = f_a \sin \Omega t$, and Newton's second law is $f_m = mdv/dt$ with $m = W/g$. When these constitutive equations are substituted in the force-balance requirement $f_m = f(t) - f_k - f_{fric}$ and v is replaced by dy/dt the result is the equation of motion

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f_a \sin \Omega t$$

(a) If the input force is considered to be the imaginary part of the complex excitation $f_a \exp(i\Omega t)$, it is appropriate to consider the steady-state response to be the imaginary part of the complex response $A\exp(i\Omega t)$ where A is the complex amplitude of the response , which is to be found by requiring the complex response to satisfy the equation of motion. Substitution of $y = A\exp(i\Omega t)$ into

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f_a \exp(i\Omega t)$$

yields

$$A = \frac{f_a}{k + i\Omega b - m\Omega^2}$$

(b) (i) At very low frequencies $A \to f_a/k$.

(ii) At very high frequencies $A \to -f_a/m\Omega^2$.

(iii) The magnitudes of (b) and (c) become equal when $\Omega^2 \to \Omega^2_{break} = k/m = \omega_o^2$.

(c) In terms of the behavioral parameters ω_o and ζ defined by

$$\omega_o^2 = \frac{k}{m} \qquad \text{and} \qquad 2\zeta\omega_o = \frac{b}{m}$$

the ratio Z of $A(\Omega)$ to $A(0)$,

$$Z = \frac{A(\Omega)}{A(0)} = \frac{1}{(1 - \frac{\Omega^2}{\omega_o^2}) + i2\zeta\frac{\Omega}{\omega_o}}$$

is a convenient dimensionless form of the complex amplitude A. The engine displacement amplitude has its peak magnitude when $|Z|^2$ is a maximum. Now

$$|Z|^2 = \frac{1}{(1 - \frac{\Omega^2}{\omega_o^2})^2 + 4\zeta^2\frac{\Omega^2}{\omega_o^2}}$$

and its maximum occurs when its derivative with respect to Ω^2/ω_o^2 vanishes; *i.e.*, when

$$\frac{d|Z|^2}{d\Omega^2/\omega_o^2} = 2(1 - \frac{\Omega^2}{\omega_o^2}) + 4\zeta^2 = 0$$

The solution of this equation for Ω yields

$$\Omega_{peak} = \omega_o\sqrt{1 - 2\zeta^2}$$

(d) To evaluate $|Z(\Omega_{peak})|$, insert $\Omega^2_{peak} = \omega_o^2(1 - 2\zeta^2)$ in the preceding expression for $|Z|^2$ to get

$$|Z(\Omega_{peak})|^2 = \frac{1}{4\zeta^4 + 4\zeta^2(1 - 2\zeta^2)} = \frac{1}{4\zeta^2(1 - \zeta)^2}$$

so that

$$|Z(\Omega_{peak})| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

For comparison, note that $|Z(\omega_o)| = 1/2\zeta$. The actual peak occurs at a frequency that is smaller than the undamped natural frequency ω_o by a factor of $\sqrt{1 - 2\zeta^2}$ and the magnitude of the peak is greater than the magnitude of the response at $\Omega = \omega_o$ by the factor $1/\sqrt{1 - \zeta^2}$.

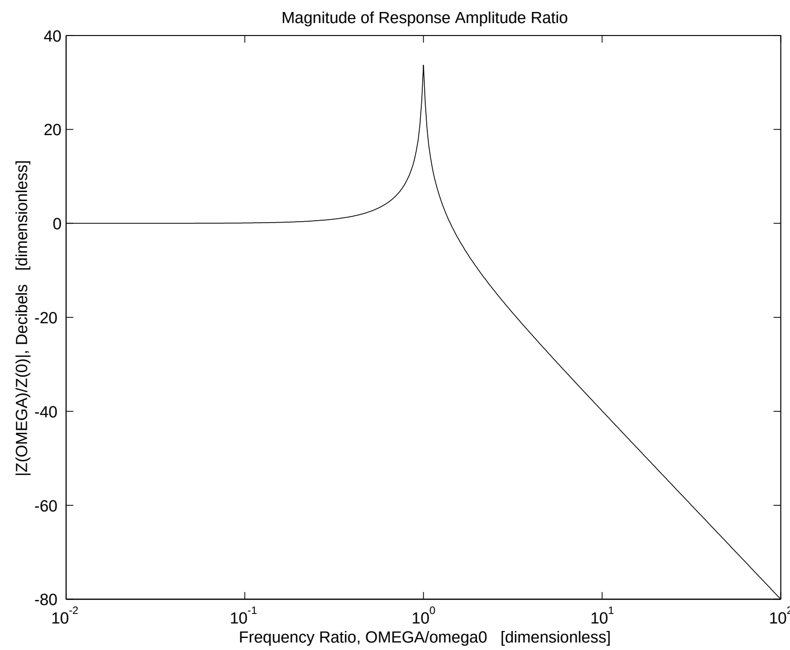

Figure 2: Bode Plot of Magnitude of $A(\Omega)/A(0)$ in dB vs. Frequency Ratio Ω/ω_o

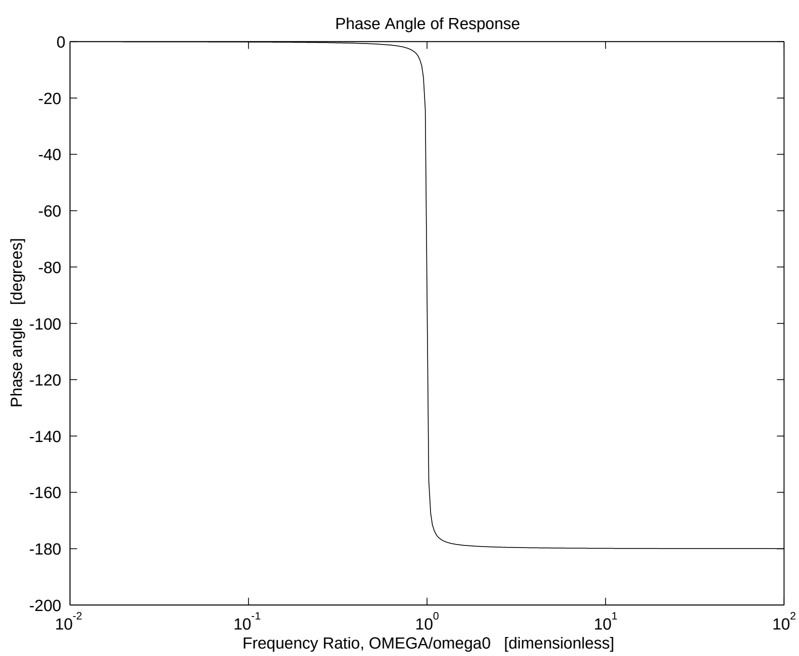

Figure 3: Bode Plot of Phase Angle vs. Frequency Ratio Ω/ω_o

(e) The Bode plots shown in Fig.2 and Fig.3 were obtained by running the following script (adapted from the Script for plotting Bode plots for 2.003 Assignment 8, Problem 1):

```
% Bode9_1.m makes Bode Plots for Problem 1 in 2.003 Assignment 9,
% in terms of the amplitude ratio Z = A(OMEGA)/ A(omega0) and the
% frequency ratio OMEGA/omega0 (OmRa = Omega Ratio).

clear variables
zeta = 0.01035;

OmRa = logspace(-2, 2, 200);
unity = ones(1,length(OmRa));
Re = unity - OmRa.^2;
Im = 2* zeta *OmRa;
Dsq = Re.^2 + Im.^2;
MagZsq = unity ./ Dsq;
dB = 10* log10(MagZsq);
phi = 57.3 * atan2( -Im , Re );

semilogx(OmRa,dB), title('Magnitude of Response Amplitude Ratio'),
xlabel('Frequency Ratio, OMEGA/omega0   [dimensionless]'),
ylabel('|Z(OMEGA)/Z(0)|, Decibels   [dimensionless] '),
pause

semilogx(OmRa, phi), title('Phase Angle of Response'),
xlabel('Frequency Ratio, OMEGA/omega0   [dimensionless]'),
ylabel('Phase angle   [degrees]')
```

2. (27 points, 5 points each for parts a, c and d, 6 points each for parts b and e)Repeat Problem 1, but with $f(t) = C\Omega^2 \sin \Omega t$ with $C = 0.005$ pound-sec^2.

(a) Substitution of $y = A\exp(i\Omega t)$ into

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = C\Omega^2\exp(i\Omega t)$$

yields

$$A = \frac{C\Omega^2}{k + i\Omega b - m\Omega^2}$$

(b) (i) At very low frequencies $A \to C\Omega^2/k$.

(ii) At very high frequencies $A \to -C/m$.

(iii) The magnitudes of (b) and (c) become equal when $\Omega^2 \to \Omega_{break}^2 = k/m = \omega_o^2$.

(c) In terms of the behavioral parameters ω_o and ζ defined by

$$\omega_o^2 = \frac{k}{m} \qquad \text{and} \qquad 2\zeta\omega_o = \frac{b}{m}$$

the ratio Z of $A(\Omega)$ to $A(\infty)$,

$$Z = \frac{A(\Omega)}{A(\infty)} = \frac{\frac{\Omega^2}{\omega_o^2}}{(1 - \frac{\Omega^2}{\omega_o^2}) + i2\zeta\frac{\Omega}{\omega_o}}$$

is a convenient dimensionless form of the complex amplitude A. The engine displacement amplitude has its peak magnitude when $|Z|^2$ is a maximum. Now

$$|Z|^2 = \frac{\frac{\Omega^4}{\omega_o^4}}{(1 - \frac{\Omega^2}{\omega_o^2})^2 + 4\zeta^2\frac{\Omega^2}{\omega_o^2}}$$

and its maximum occurs when its derivative with respect to Ω^2/ω_o^2 vanishes; *i.e.*, when

$$2\frac{\Omega^2}{\omega_o^2}[(1 - \frac{\Omega^2}{\omega_o^2})^2 + 4\zeta^2\frac{\Omega^2}{\omega_o^2}] - \frac{\Omega^4}{\omega_o^4}[-2(1 - \frac{\Omega^2}{\omega_o^2}) + 4\zeta^2] = 0$$

The solution of this equation for Ω^2/ω_o^2 yields

$$\frac{\Omega^2}{\omega_o^2} = \frac{1}{1 - 2\zeta^2}$$

so that

$$\Omega_{peak} = \frac{\omega_o}{\sqrt{1 - 2\zeta^2}}$$

(d) To evaluate $|Z(\Omega_{peak})|$, insert $\Omega_{peak}^2/\omega_o^2 = 1/(1-2\zeta^2)$ in the preceding expression for $|Z|^2$ to get

$$|Z(\Omega_{peak})|^2 = \frac{(\frac{1}{1-2\zeta^2})^2}{(1 - \frac{1}{1-2\zeta^2})^2 + \frac{4\zeta^2}{1-2\zeta^2}} = \frac{1}{4\zeta^2(1 - \zeta)^2}$$

so that

$$|Z(\Omega_{peak})| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

Comparing this result with that of Problem 1, we note that the actual peak here occurs at a frequency that is *greater* than the undamped natural frequency ω_o by a factor of $1/\sqrt{1 - 2\zeta^2}$, but the magnitude of the peak is the same. It is greater than the magnitude of the response at $\Omega = \omega_o$ by the same factor $1/\sqrt{1 - \zeta^2}$.

(e) The Bode plots shown in Fig.4 and Fig.5 were obtained by running the following script (adapted from the Script for Problem 1):

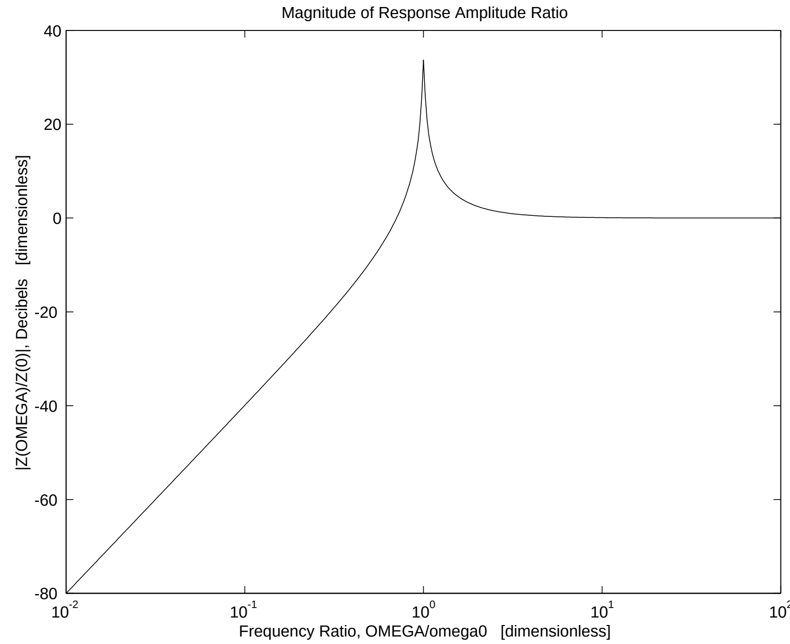

Figure 4: Bode Plot of Magnitude of $A(\Omega)/A(0)$ in dB vs. Frequency Ratio Ω/ω_o

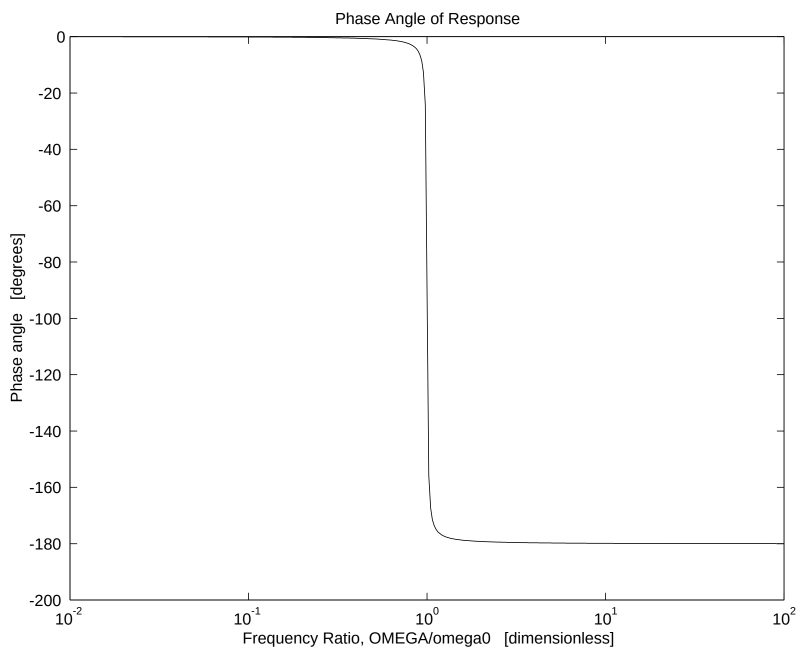

Figure 5: Bode Plot of Phase Angle vs. Frequency Ratio Ω/ω_o

```
\begin{verbatim}
% Bode9_2.m makes Bode Plots for Problem 2 in 2.003 Assignment 9,
% in terms of the amplitude ratio Z = A(OMEGA)/ A(omega0) and the
```

```
% frequency ratio OMEGA/omega0

clear variables
zeta = 0.01035;

OmRa = logspace(-2, 2, 200);
unity = ones(1,length(OmRa));
Re = unity - OmRa.^2;
Im = 2* zeta *OmRa;
Dsq = Re.^2 + Im.^2;
Nsq = OmRa.^4
MagZsq = Nsq ./ Dsq;
dB = 10* log10(MagZsq);
phi = 57.3 * atan2( -Im , Re );

semilogx(OmRa,dB), title('Magnitude of Response Amplitude Ratio'),
xlabel('Frequency Ratio, OMEGA/omega0   [dimensionless]'),
ylabel('|Z(OMEGA)/Z(0)|, Decibels   [dimensionless] '),
pause

semilogx(OmRa, phi), title('Phase Angle of Response'),
xlabel('Frequency Ratio, OMEGA/omega0   [dimensionless]'),
ylabel('Phase angle   [degrees]')
```

3. (22 points, 4 points for part a, 6 points for part b, 12 points for part c)The equation of motion

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t) = f_a \sin \Omega t$$

obtained in Problem 1 can be rewritten in the standard form for state-determined systems by taking $y(t)$ and $v(t) = dy/dt$ as state varibles and writing the state equations in matrix form

$$\frac{d}{dt}\left\{ \begin{array}{c} y \\ \\ v \end{array} \right\} = \left[\begin{array}{cc} 0 & 1 \\ \\ -\frac{k}{m} & -\frac{b}{m} \end{array} \right] \left\{ \begin{array}{c} y \\ \\ v \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ \\ 1 \end{array} \right\} \frac{f_a}{m} \sin \Omega t$$

One of the desired outputs is the state variable $y(t)$. The other outputs desired are $P_{in} = f(t) * v(t)$; $P_{diss} = bv * v$; and $P_{stored} = P_{in} - P_{diss}$. The values of the physical parameters are:

$$\begin{array}{lcl} m = 200 \text{ pounds} & = & 90.7 \text{ kg} \\ k = 18,000\text{pounds/inch} & = & 3.152e6 \text{ Newton/meter} \\ b = 2 \text{ pounds/inch/second} & = & 350 \text{ Newtons/meter/second} \\ f_a = 2 \text{ pounds} & = & 8.90 \text{ Newtons} \\ \Omega_{break}/3 = 62.1 \text{ rad/sec} & = & 62.1 \text{ rad/sec} \\ \Omega_{break} = 186.4 \text{ rad/sec} & = & 186.4 \text{ rad/sec} \\ 3\Omega_{break} = 559 \text{ rad/sec} & = & 559 \text{ rad/sec} \end{array}$$

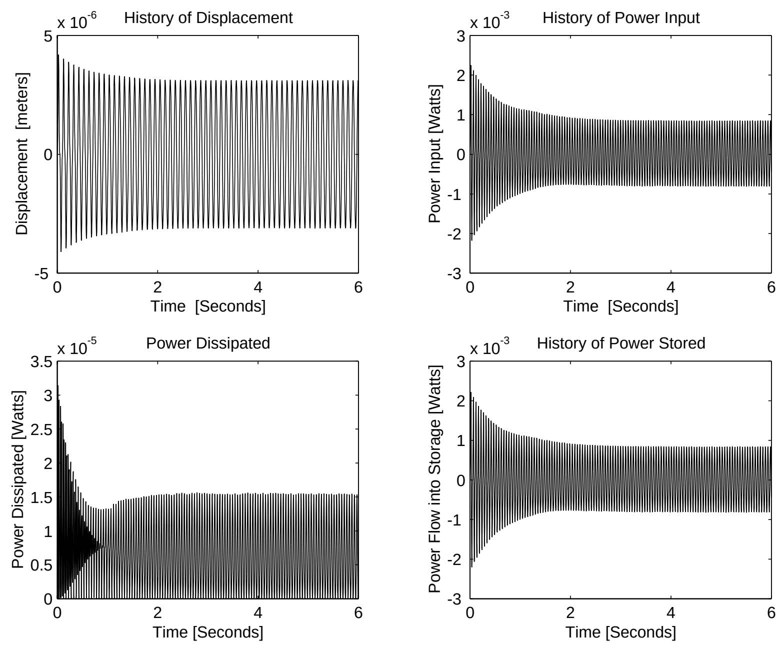

Figure 6: Engine Start-Up when $\Omega = \Omega_{break}/3$

The plots shown in Figs. 6-8 were obtained by running the following MATLAB scripts (adapted from 'POS.m' and 'pl_on_spr.m' used in Assignment 5. The input script is called 'EngStart'

```
% 'EngStart.m', a MATLAB script for Problem 3 in 2.003 Assignment 9. Produces plots o
%   (i) position vs. time
%  (ii) input power vs. time
% (iii) dissipated power vs. time
%  (iv) power flow into storage vs. time
% for the response of an engine block, with mass m, stiffness k,
% and damping parameter b, when the system starts from initial conditions
% y = 0 and v = 0 under the action of a suddenly applied sinusoidal force
% fa sin OMEGA t, at t = 0.

clear variables
global m k b fa Om
% Input parameters
m = 90.7; %input('Enter the mass "m" in kilograms ');
k = 3.152e6; %input('Enter the stiffness "k" in Newtons/meter ' );
b = 350; %input('Enter the damping constant "b" in kilograms/sec ');
fa= 8.90; %input('Enter the magnitude "fa" of the suddenly applied force in Newtons '
Om= input('Enter the frequency OMEGA of the suddenly applied force in rad/sec ');
% Input initial conditions.
y0= 0;
v0= 0;
```

```
tspan = input('Enter the duration "T" of the desired time history, in seconds ');
X0 = [ y0 ; v0 ];
% Integrate equations of motion
[t,X] = ode45('EqEngStart', tspan, X0);
% Construct power variables
Pin = fa * sin(Om*t) .* X(:,2);
Pdiss = b*X(:,2).^2;
Pstored = Pin - Pdiss;
% Plot results
subplot(221),plot(t,X(:,1)), title('History of Displacement'),
xlabel('Time  [Seconds]'), ylabel('Displacement  [meters]'), pause

subplot(222),plot(t,Pin), title('History of Power Input'),
xlabel('Time  [Seconds]'), ylabel('Power Input [Watts]'), pause

subplot(223),plot(t, Pdiss), title('Power Dissipated'),
xlabel('Time [Seconds]'), ylabel('Power Dissipated [Watts]'), pause

subplot(224),plot(t, Pstored), title('History of Power Stored'),
xlabel('Time [Seconds]'), ylabel('Power Flow into Storage [Watts]')
```

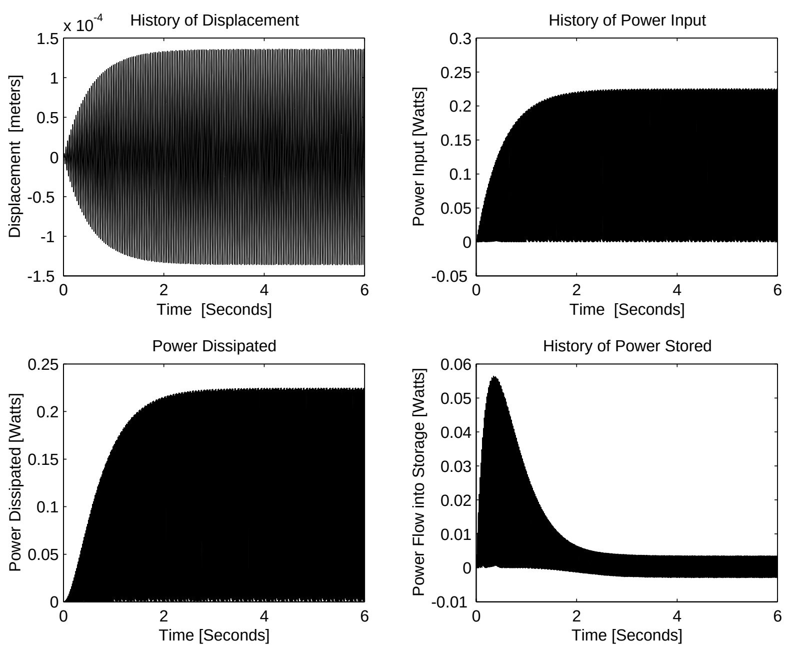

Figure 7: Engine Start-Up when $\Omega = \Omega_{break}$

The equation script which is called by 'EngStart' follows:

```
% 'EqEngStart.m'  Provides equation of motion for initial transient
```

```
% to be integrated by script 'EngStart.m'
function Xdot = EqEngStart(t,X)

global m k b fa Om

Xdot = [ 0  1 ; -k/m -b/m ]*X + [ 0 ; fa/m*sin(Om*t)];
```

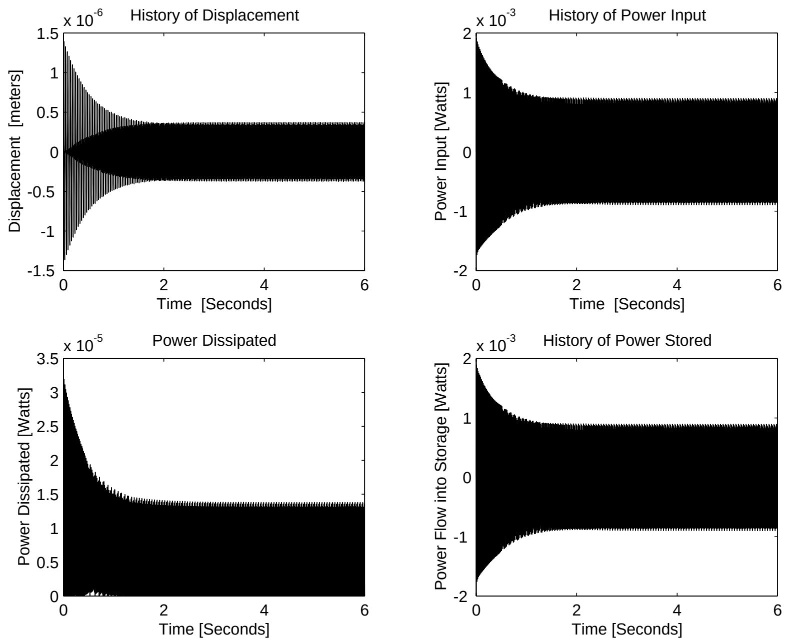

Figure 8: Engine Start-Up when $\Omega = 3\Omega_{break}$

4. (24 points, 12 points for each part)Torsional vibration of coupler shaft between engine and propellor. Moment of inertia of propellor is estimated as $I = mL^2/12$, where m is the mass of an aluminum rod, 2 inches (0.0508 meters) in diameter and six feet (1.829 meters) in length. The density of aluminum is 2720 kg/m^3, so

$$m = \rho\pi r^2 L = 2720\pi(0.0254)^2(1.829) = 10.08 \text{ kg}$$

and

$$I = \frac{mL^2}{12} = \frac{(10.08)(1.829)^2}{12} = 2.811 \text{ kg-m}^2$$

(a) The input is the given angular displacement of the engine end of the shaft

$$\theta_e = \Omega t + \epsilon \sin\frac{\Omega}{2}t$$

Take the output to be the angular displacement θ_p and angular velocity $\omega_p = d\theta_p/dt$ of the propellor end of the shaft. Assume that the that the torque T_{shaft}

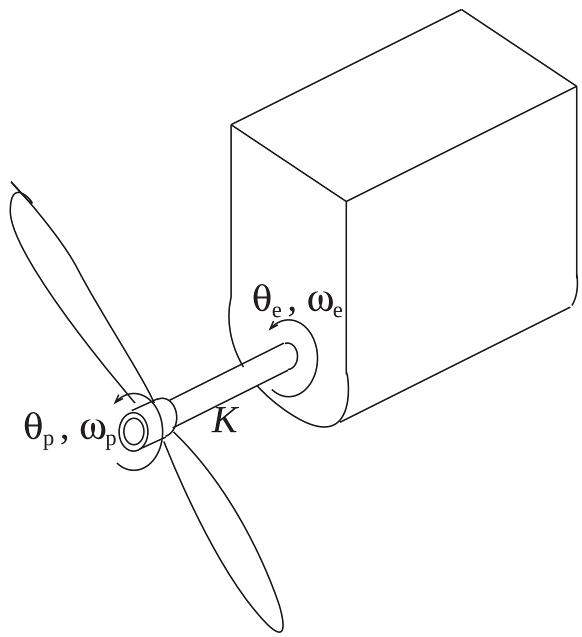

Figure 9: Torsional Vibration of Propellor Shaft

transmitted to the propellor from the shaft and the torque T_{fric} applied to the propellor by the air resistance can be modelled by the *linear* relations

$$T_{shaft} = K(\theta_p - \theta_e) \qquad \text{and} \qquad T_{fric} = B\theta_p$$

The torque balance requirement is

$$-T_{shaft} - T_{fric} = T_I = I\frac{d\omega_p}{dt}$$

These relations can be combined in the standard form for a state-determined system

$$\frac{d}{dt}\left\{\begin{array}{c}\theta_p\\ \\ \omega_p\end{array}\right\} = \left[\begin{array}{cc}0 & 1\\ \\ -K/I & -B/I\end{array}\right]\left\{\begin{array}{c}\theta_p\\ \\ \omega_p\end{array}\right\} + \left\{\begin{array}{c}0\\ \\ 1\end{array}\right\}\frac{K}{I}(\Omega t + \epsilon\sin\frac{\Omega}{2}t)$$

or in the form of a single second-order equation

$$I\frac{d^2\theta_p}{dt^2} + B\frac{d\theta_p}{dt} + K\theta_p = K(\Omega t + \epsilon\sin\frac{\Omega}{2}t)$$

The input motion consists of a uniform rotation at angular rate Ω plus a sinusoidal oscillation at frequency $\Omega/2$. Because the equation is *linear* the steady-state responses to these two inputs can be obtained separately and superposed to obtain the complete solution. For the uniform rotation, try a response of the form

$$\theta_{p,unif} = \Omega t + \text{Constant}$$

and find that it satisfies

$$I\frac{d^2\theta_{p,unif}}{dt^2} + B\frac{d\theta_{p,unif}}{dt} + K\theta_{p,unif} = K\Omega t$$

if the Constant $= -B\Omega/K$. this means that, after an initial transient, uniform engine speed results in a uniform propellor speed with a constant twist in the shaft just large enough for the shaft torque to balance the friction torque. For the sinusoidal oscillation, the response $\theta_{p,osc} = \psi(t)$ must satisfy the equation

$$I\frac{d^2\psi}{dt^2} + B\frac{d\psi}{dt} + K\psi = K\epsilon \sin\frac{\Omega}{2}t$$

A solution is sought in the form

$$\psi = \mathrm{Im}\{A\ \exp(i\frac{\Omega}{2}t)\}$$

The complex amplitude A must satisfy the relation

$$\frac{A}{\epsilon} = \frac{\frac{K}{I}}{\frac{K}{I} - \frac{\Omega^2}{4} + i\frac{B\Omega}{2I}} = \frac{\omega_o^2}{\omega_o^2 - \frac{\Omega^2}{4} + i\zeta\omega_o\Omega}$$

where

$$\omega_o^2 = \frac{K}{I} \qquad \text{and} \qquad 2\zeta\omega_o = \frac{B}{I}$$

This result is very similar to the result obtained in Problem 1, with the exception that here the firing frequency $\Omega/2$ plays the role of the driving frequency Ω in Problem 1. The ratio of the complex amplitude at frequency $\Omega/2$ to the response when $\Omega \to 0$ is

$$Z = \frac{A(\Omega/2)}{A(0)} = \frac{1}{[1 - \frac{(\Omega/2)^2}{\omega_o^2}] + i2\zeta\frac{(\Omega/2)}{\omega_o}}$$

In Problem 1, it was shown that the peak value of $|Z|$ is

$$|Z|_{peak} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

and that the peak amplitude is realized when the forcing frequency is

$$\Omega_{peak} = \omega_o\sqrt{1-2\zeta^2}$$

(b) To use the results of Problem 1, we need the ratio, $|Z|_{peak}$, of the peak response to the zero-frequency response and the magnitude of the forcing frequency which produces the peak response. Here we know that peak response occurs when the engine speed is 2200 rpm. The corresponding forcing frequency is $\Omega/2 = 1100$ rpm or 115.2 rad/sec. We do not know the zero-frequency response, but we do know that the ratio of the peak response, at 2200 rpm, to the response at 500 rpm is 4. We also know from Fig.2 that the response magnitude for low frequencies, well below the natural frequency, is not much greater than the zero-frequency response. Therefor we can take the given ratio of 4 as an estimate of the ratio $|Z|_{peak}$. With these estimates, we can use the results of Problem 1 to obtain estimates of the behavioral parameters ω_o and ζ.

$$\omega_o = \frac{\Omega_{peak}}{\sqrt{1-2\zeta^2}} = \frac{115.2}{\sqrt{1-2\zeta^2}} \qquad \text{and} \qquad |Z|_{peak} = 4 = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

The solution of these equations yields the estimates

$$\zeta = 0.1260 \qquad \text{and} \qquad \omega_o = 117.1 \text{ rad/sec}$$

from which follow

$$K = I\omega_o^2 = 2.811(117.1)^2 = 38,500 \text{ N-m/radian}$$

and

$$B = 2\zeta\omega_o I = 2(0.126)(117.1)(2.811) = 83.0 \text{ N-m/rad/sec}$$

as estimates of the model parameters.