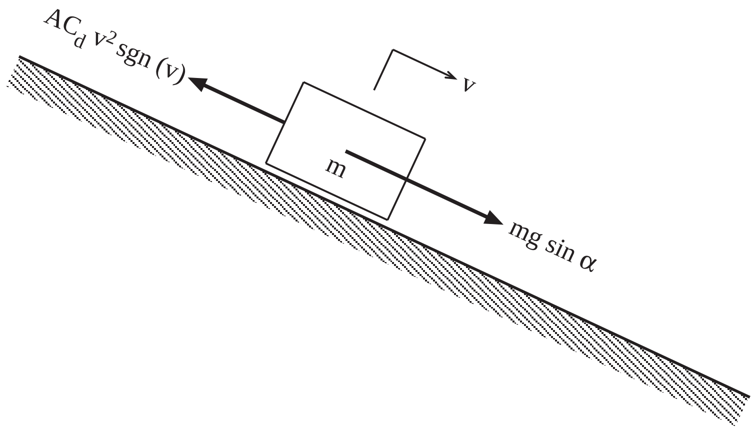
2.003 Fall 1999 Solution of Homework Assignment 2

1.(a) Replace $f_{viscous}$ by f_{drag} to get the free body diagram of the car mass acted on by the component of the weight, and the retarding *form-drag* force, shown below.



The equation of motion for the model is

$$m\frac{dv}{dt} = mg\sin\alpha - AC_d v^2 \mathrm{sgn}\, v$$

(b) The dimensions of the force f_{drag} are $[F] = [M][L]/[T^2]$; the dimensions of the area A are $[L^2]$; the dimensions of the velocity squared $v^2 = [L^2]/[T^2]$; and the *signum* function is dimensionless. The dimensions of the parameter C_d are

$$[C_d] = \left[\frac{f_{drag}}{Av^2 \mathrm{sgn} v}\right] = \left[\frac{ML/T^2}{(L^2)(L^2/T^2)}\right] = [M/L^3]$$

In SI units, the dimensions of C_d are $\mathrm{kg/m^3}$.

(c) On a horizontal track $\alpha = 0$, and the only horizontal force acting on the mass is the retarding form-drag force. Starting with a positive velocity v_o at $t = 0$, the mass will slow down, but will not reverse direction; *i.e.,* $v(t)$ will be non-negative, so the equation of motion can be written

$$m\frac{dv}{dt} = -AC_d v^2$$

The initial condition is $v = v_o$ at $t = 0$. The differential equation can be integrated by separating the variables v and t

$$\int_{v_o}^{v} \frac{dv}{v^2} = -\int_0^t \frac{AC_d}{m} dt$$

$$\frac{-1}{v} + \frac{1}{v_o} = -\frac{AC_d}{m}t$$

The time history of the speed of the car is obtained by solving the above equation for $v(t)$.

$$v(t) = \frac{v_o}{1 + \frac{AC_d v_o}{m}t}$$

Note that $v(t)$ decreases smoothly with time, and asymptotically approaces the limit $v = 0$ as t approaces infinity.

2. The differential equation for the model in which a car of mass m descends a very long inclined track with angle α under the influence of gravity and form drag is

$$m\frac{dv}{dt} = mg\sin\alpha - AC_d v^2 \text{sgn}\, v$$

as given at the beginning of Problem 1.

(a) At the terminal velocity v_{ss} there is no longer any acceleration ($dv/dt = 0$), and the drag force is equal and opposite to the accelerating force component, so that

$$v_{ss}^2 = \frac{mg\sin\alpha}{AC_d}$$

(b) If $m = 0.1418$ kg, $\alpha = 10.81$ degrees $= \sin^{-1}(3/16)$, and v_{ss} =8.60 m/s, then the product of the frontal area A and the drag coefficient C_d is

$$AC_d = \frac{mg\sin\alpha}{v^2} = \frac{(0.1418)(9.81)(3/16)}{(8.60)^2} = 0.00353 \text{ kg/m}$$

(c) To use the scripts 'car2.m' and 'car_form.m' to determine the time T for V to reach $(0.999)(8.60) = 8.5914$ m/s fix the inputs of $m = 0.1418$ kg, $\alpha = 10.81$ deg, and $AC_d = 0.00353$ kg/m^{3}n and then call 'car2' and input various guesses for T, iterating toward "Final speed = 8.5914 m/s". For example:

Try T = 10 secs, get V = 8.362 m/s

Try T = 20 secs, get V = 8.594 m/s

Try T = 19 secs, get V = 8.5919 m/s

Try T = 18.9 secs, get V = 8.5917 m/s

Try T = 18.8 secs, get V = 8.5915 m/s

Try T = 18.7 secs, get V = 8.5913 m/s

Try T = 18.75 secs, get V = 8.5914 m/s (close enough)

This result is not very accurate, because of round-off error. The value 0.00353 kg/m obtained in (b) above for the parameter aC_d was rounded off to three significant figures. If we assume that AC_d is exactly equal to 0.00353 kg/m, and work backwards we find that v_{ss} is not exactly equal to 8.60 m/s but is actually 8.5958 m/s correct to five significant figures. If we repeat the exercise we find that at T = 17.45 sec, the final speed is 8.5872 m/s which is 99.9 % of 8.5958 m/s. Both of the previous results assume that there are no errors in the integration carried out by MATLAB. MATLAB is pretty good, but it does make small errors which can accumulate over a long integration. If we let it run for long times T, we find that throughout the range from $T = 35$ secs to $T = 70$ secs, the final speed is essentially constant (to five significant figures) at 8.5970 m/s; *i.e.,* MATLAB disagrees, in the fourth significant figure, with both $v_{ss} = 8.6000$ m/s and $v_{ss} = 8.5958$ m/s.

(d) In Problem 4(b) of Assignment 1, it was found that with viscous friction the time to reach 99.9% of the terminal velocity was in the neighborhood of 28.5 to 32.5 seconds. Here, with form drag, the same speed is reached in about 17.45 to 18.75 seconds . With form drag (quadratic speed dependence) the terminal velocity is approached more quickly than is the case for viscous friction (linear speed dependence). The time histories of velocity for the two models are shown in the Fig. 2(d)-1.

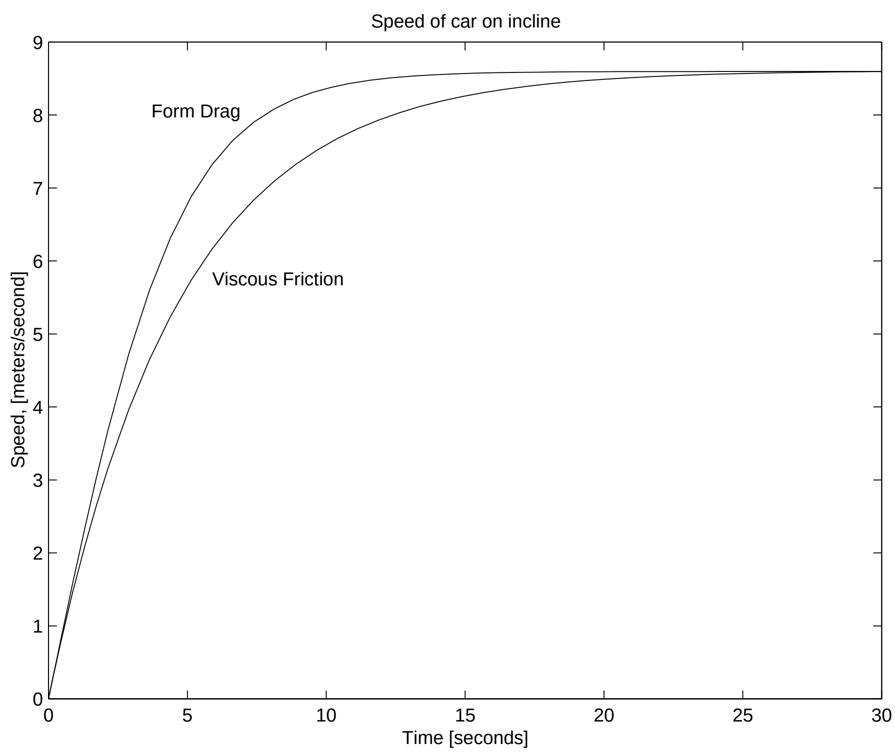


Figure 1: Fig. 2(d)-1

To explain why a car with quadratic drag approaches the terminal velocity more rapidly than a car with linear drag, compare the constitutive equations for the retarding forces as shown in Fig. 2(d)-2. At all speeds below the common terminal velocity there is less retardation with form drag than there is for viscous friction. This means that, when both cars have the same velocity, the car with form drag accelerates faster than the car with viscous friction. Now the the slopes of the velocity curves in Fig. 2(d)-1 represent the accelerations of the cars. Thus at each speed, the form drag curve will have a steeper slope than the viscous friction curve. As a consequence the form drag curve must approach the terminal velocity more quickly than the viscous friction curve.

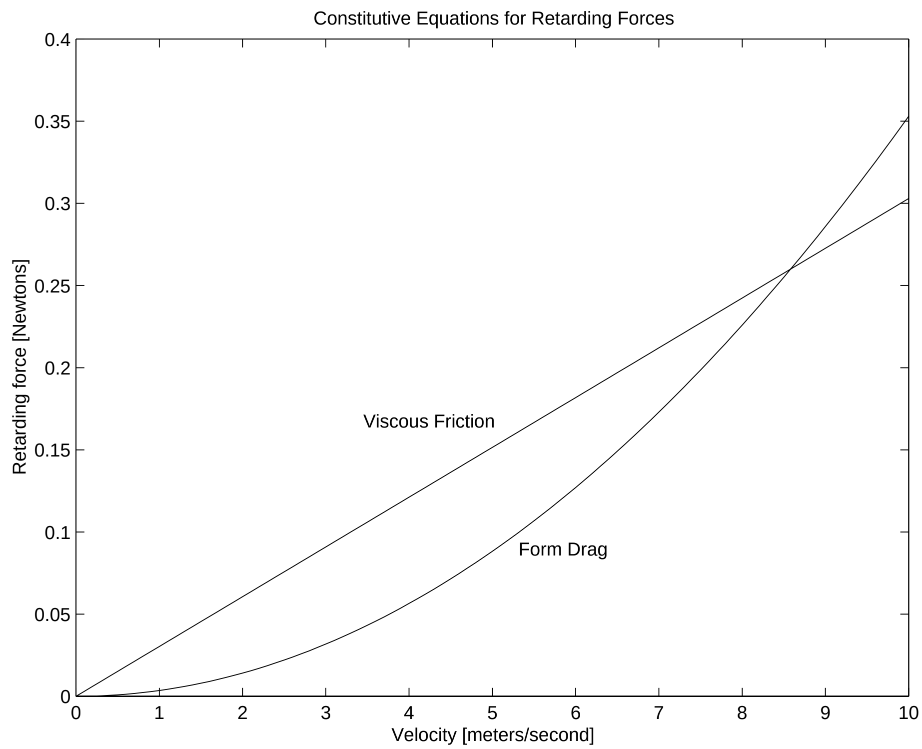
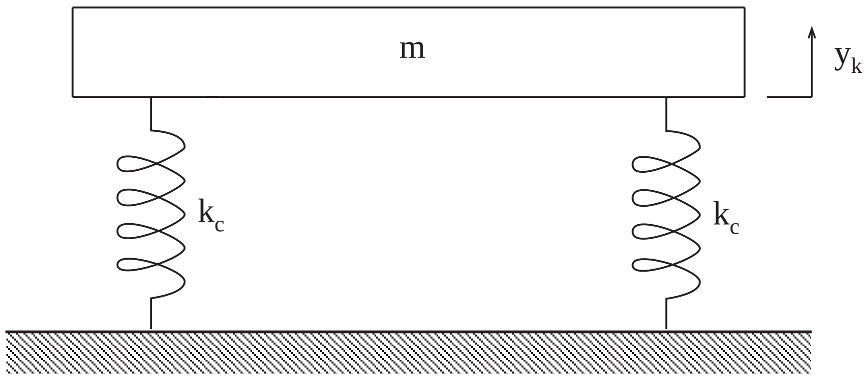


Figure 2: Fig. 2(d)-2

NOTE re MATLAB technique: To plot two graphs in the same Figure, as in Fig. 2(d)-1. Run 'car' to obtain a plot of viscous friction case. Then, in the MATLAB Command Window, type “hold on”. (this keeps the plot in the Figure screen). Next run 'car2' (with the same time interval T). The plot for the form-drag case then appears on top of the existing plot. To print captions such as ”Viscous Friction” on the interior of the plot, type, in the MATLAB Command Window, “gtext('Viscous Friction')”. When you hit 'return' the cursor on the Figure turns into a giant cross which can be moved by the mouse. Then when you click the mouse the caption appears on the Figure starting from the intersection point of the cross. You can make as many additional plots as you wish, as long as 'hold' is 'on'. To return to normal one-at-a-time plots, type “hold off” in the MATLAB Command Window.

3. The displacement y_k measures the vertical extension of the four springs from



the equilibriium configuration where the weight of the plate mg is balanced by initial compressive forces in the springs. The excess tensile force in each spring due to a displacement y_k is $f_{ki} = k_c y_k \quad (i = 1, \cdots, 4)$. The total vertical spring force is $f_k = 4f_{ki} = 4k_c y_k = ky_k$. The constitutive equation for the mass is Newton's law $f_m = mdv_m/dt$. When friction is absent and there is no external load, the force balance equation is simply $-f_k = f_m$, or

$$-ky_k = m\frac{dv_m}{dt}, \quad \text{or} \quad -ky_k = m\frac{dv_k}{dt}$$

on using the geometric compatibility condition $v_m = v_k$. The standard form of the differential equation (See Lecture notes of 9/15/99) is

$$\frac{d^2 y_k}{dt^2} + (\frac{k}{m})y_k = 0$$

Guess a solution of the form $y_k = A\exp(\lambda t)$, which leads to the characteristic equation

$$\lambda^2 + \omega_o^2 = 0$$

where $\omega^2 = k/m$. The roots of the characteristic equation are $+i\omega_o$ and $-i\omega_o$, and the general solution of the differential erquation is

$$y_k = A\exp(i\omega_o t) + B\exp(-i\omega t)$$

An alternative form of the general solution can be obtained by introducing *Euler's formula* $\exp(i\omega_o t) = \cos\omega_o t + i\sin\omega_o t$.

$$y_k = C_1 cos\omega_o t + C_2 \sin\omega_o t$$

These two forms of solution are equivalent if the constants of integration are related as follows:

$$\begin{aligned} C_1 &= A + B \qquad A = \frac{1}{2}(C_1 + C_2) \\ C_2 &= A - B \qquad B = \frac{1}{2}(C_1 - C_2) \end{aligned}$$

The given initial conditions are $y_k(0) = 0$ and $dy_k/dt(0) = v_o$. Using the trigonometric form of the general solution, we have

$$y_k = C_1 \cos \omega_o t + C_2 \sin \omega_o t \qquad \text{and} \qquad \frac{dy_k}{dt} = -C_1 \omega_o \sin \omega_o t + C_2 \omega_o \cos \omega_o t$$

Setting $t = 0$ yields

$$C_1 = 0 \qquad \text{and} \qquad C_2 = \frac{v_o}{\omega_o}$$

so the displacement history of the plate, starting from rest at the equilibrium position with the initial velocity v_o is

$$y_k(t) = \frac{v_o}{\omega_o} \sin \omega_o t$$

4. The unloaded plate system here is the same as that considered in Problem 3. The mass of the plate is m and the effective stiffness of the four springs is k. The natural frequency of the unloaded system is ω_o where $\omega^2 = k/m$. When the book of mass $m/2$ is attached to the plate, the resulting system has mass $3m/2$ and stiffness k, and its natural frequency of free oscillation is ω_1 where $\omega_1^2 = 2k/3m$. The oscillations in both Case I and Case II involve motions with the book attached to the plate. The difference between Cases I and II lies in the initial conditions, not in the basic system.

(a) The ratio of frequencies of Case I oscillations to unloaded plate oscillations is $\omega_1/\omega_o = \sqrt{2/3} = 0.816$.

(b) The ratio of frequencies of Case II oscillations to unloaded plate oscillations is $\omega_1/\omega_o = \sqrt{2/3} = 0.816$.

(c) In the (book + plate) system there is an additional gravity load of $mg/2$, so the equilibrium position of the (book + plate) system is beneath the equilibrium position of the unloaded plate by a distance of $\Delta = mg/2k$.

(d) In Case I the system starts from rest at the equilibrium position of the unloaded plate. Viewed from the equilibrium position of the (book + plate) system, the initial displacement $y_k(0) = \Delta$, and the initial velocity is $dy_k/dt(0) = 0$.

(e) In Case II there is an initial velocity $dy_k/dt(0) = -v_1$ imparted by the impact of the book upon the plate. This initial velocity can be determined by applying conservation of linear momentum to the impact. Immediately before the impact the plate has no momentum while the book has a downward momentum of $1/2mv_o$. Immediately after the impact the (book + plate) has the downward momentum $3mv_1/2$. For these two momenta to be equal it is necessary that $v_1 = v_o/3$. The initial conditions for Case II are thus: $y_k(0) = \Delta$ and $dy_k/dt(0) = -v_o/3$.

(f) The general solution for both Case I and Case II is the same as the general solution for Problem 3, except that ω_o is everywhere replaced by ω_1. For Case I, the initial conditions require

$$y_k(0) = \Delta = C_1 \qquad \text{and} \qquad \frac{dy_k}{dt}(0) = 0 = C_2 \omega_1$$

so that

$$y_k(t) = \Delta \cos \omega_1 t$$

where $\Delta = mg/2k$, and $\omega_1^2 = 2k/3m$.
For Case II, the initial conditions require

$$y_k(0) = \Delta = C_1 \qquad \text{and} \qquad \frac{dy_k}{dt}(0) = -v_1 = C_2\omega_1$$

so that

$$y_k(t) = \Delta \cos \omega_1 t - \frac{v_1}{\omega_1} \sin \omega_1 t$$

where $\Delta = mg/2k$, $v_1 = v_o/3$, and $\omega_1^2 = 2k/3m$