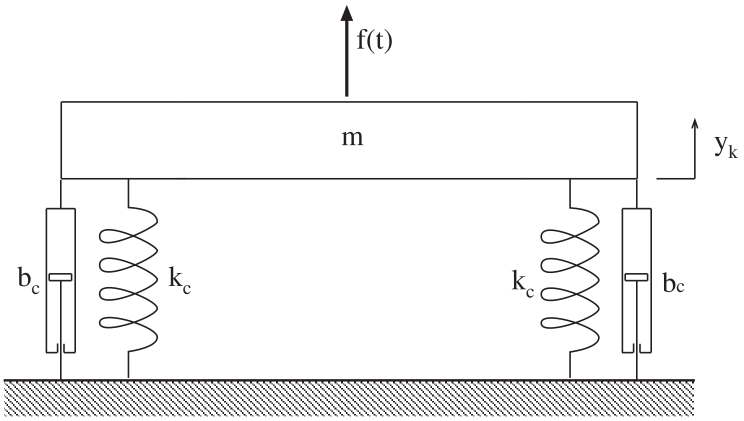
2.003 Fall 1999 Solution of Homework Assignment 3

1. The rigid plate of mass m is supported by four springs, each of stiffness k_c and damping parameter b_c. If y_k is measured from the equilibrium position where



the weight of the plate is balanced by the initial compession of the springs, the constitutive equations are

$$f_m = m\frac{dv_m}{dt} \qquad f_{ki} = k_c y_{ki} \qquad f_{bi} = b_c v_{bi} \qquad (i = 1, \cdots, 4)$$

Geometric compatibility requires that, for $(i = 1, \cdots, 4)$,

$$y_{ki} = y_k \qquad v_m = v_{bi} = \frac{dy_{ki}}{dt} = \frac{dy_k}{dt}$$

Because there are four springs,

$$f_k = \sum_1^4 f_{ki} = 4k_c y_k = k y_k \qquad \text{and} f_b = \sum_1^4 f_{bi} = 4b_c\frac{dy_k}{dt} = b\frac{dy_k}{dt}$$

Finally force balance requires that

$$f(t) - f_k - f_b = f_m$$

which leads to the differential equation

$$m\frac{d^2y_k}{dt^2} + b\frac{dy_k}{dt} + k y_k = f(t)$$

When the behavioral parameters ω_o and ζ defined by

$$\omega_o^2 = \frac{k}{m} \qquad 2\zeta\omega_o = \frac{b}{m} \quad \text{or} \quad \zeta = \frac{b}{\sqrt{4mk}}$$

are introduced, the equation takes the form

$$\frac{d^2y_k}{dt^2} + 2\zeta\omega_o\frac{dy_k}{dt} + \omega_o^2 y_k = \frac{f(t)}{m}$$

(a) The desired behavioral parameters are $\omega_o = 5Hz(2\pi \text{ rad/cycle}) = 31.4$ rads/sec and (i) $\zeta = 0.1$, (ii) $\zeta = 0.5$, (iii) $\zeta = 1.0$, and (iv) $\zeta = 1.5$. With the plate mass fixed at 5 pounds, or $5(0.4536) = 2.268$ kilograms, the spring constant k must also be fixed at

$$k = m\omega_o^2 = 2.268(31.4)^2 = 2236 Newtons/meter$$

in order to keep the undamped natural frequency, given by $\omega_o^2 = k/m$, equal to 5 Hz. To provide the desired values of ζ, the damping parameter b must be chosen to satisfy

$$\zeta = \frac{b}{2\sqrt{km}} = \frac{b}{2m\omega_o} \qquad \text{or} \qquad b = 2\zeta m\omega_o = 2(2.268)(31.4)\zeta = 142.4\zeta$$

The b-values for the four cases are: (i) 14.24 kg/sec; (ii) 71,2 kg/sec; (iii) 142.4 kg/sec; (iv) 213.6 kg/sec. The decay time constants, $\tau = 1/\zeta\omega_0$, for the first three cases are: (i) 0.318 secs: (ii) 0.0637 secs; (iii) 0.0318 secs. In Case (iv) the roots of the characteristic equation are $\lambda = \omega_o(-\zeta \pm \sqrt{\zeta^2 - 1})$. The time constants are the negative reciprocals of the λ-values, so the longest time constant is the negative reciprocal of the smallest λ

$$\tau_{longest} = -\left[31.4(-1.5 + \sqrt{(1.5)^2 - 1})\right]^{-1} = 0.0834 secs$$

The duration of the time histories in the four cases are 5 times the corresponding time constants: (i) 1.59 secs; (ii) 0.319 secs; (iii) 0.159 secs; (iv) 0.417 secs. The damped natural frequency ω_d, for cases with $0 < \zeta < 1$, is given by

$$\omega_d = \omega_o\sqrt{1 - \zeta^2}$$

For Case (i) $\omega_d = 31.26$ rad/sec for $\zeta = 0.1$, and for case (ii) $\omega_d = 27.21$ rad/sec. For Cases (iii) and (iv), the eigenvalues are purely real, and $\omega_d = 0$.

The data for the four cases in Part (a) are assembled in the Table below.

Case	ζ	k [N/m]	b [N/m/sec]	τ [sec]	5τ [sec]	ω_d [r/s]	$\zeta\omega_o$
(i)	0.1	2239	14.25	0.3183	1.592	31.26	3.124
(ii)	0.5	2239	71.25	0.06365	0.3183	27.21	15.62
(iii)	1.0	2239	142.5	0.03183	0.1592	0.0	31.42
(iv)	1.5	2239	213.8	0.08332	0.4166	0.0	47.13

In addition to these data, every case has $m = 2.268$ kg, $f_a = (5)(4.448) = 22.24$ Newtons, and the initial conditions, $y_k(0) = 0$ and $v_k(0) = 0$.

The time histories are obtained by inputting the above data into the MATLAB scripts. When $0 < \zeta < 1$, Cases (i) and (ii), the script to be used is 'MassSprgDmpr1.m'. When $\zeta = 1$, Case (iii), the script to be used is 'MassSprgDmpr2.m', and when $1 < \zeta$, Case (iv), the script to be used is 'MassSprgDmpr3.m'.

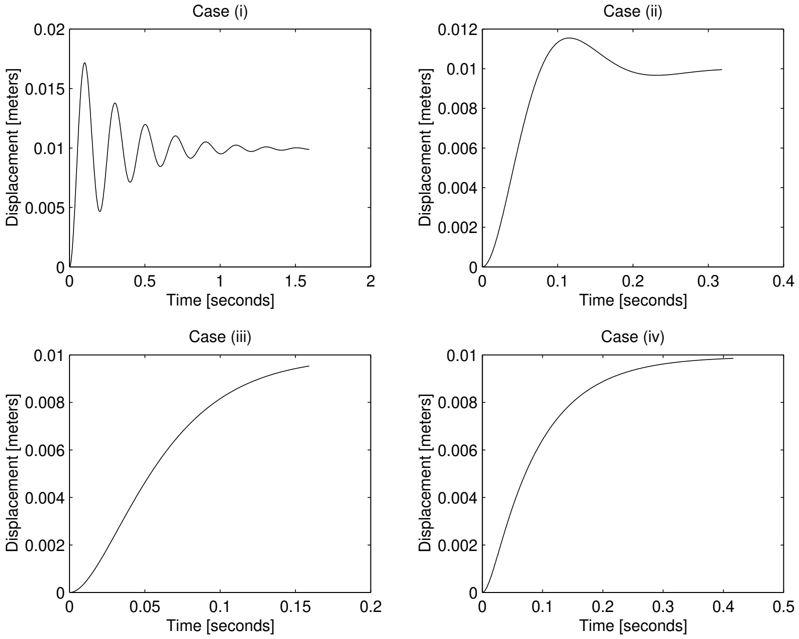


Figure 1: Part (a)

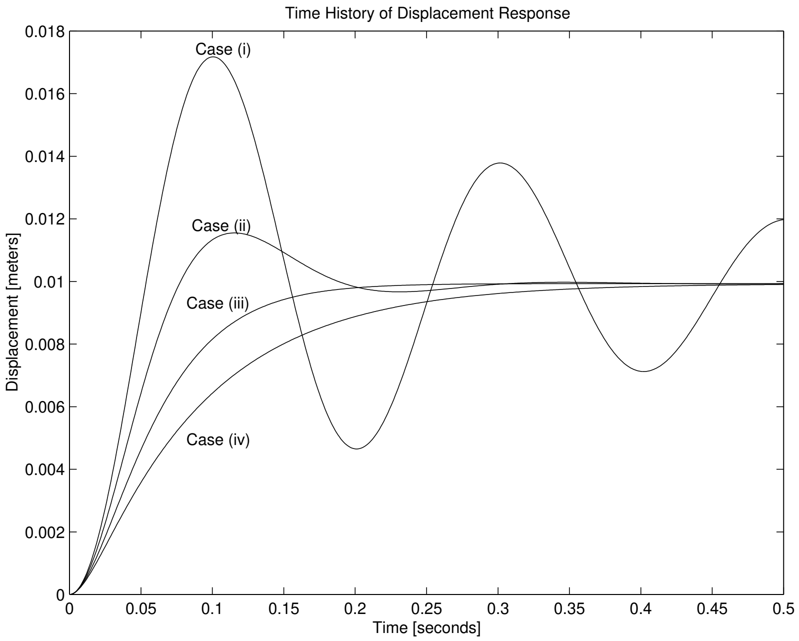


Figure 2: Part (a)

The results are displayed in Fig. 1. In Fig.1 the duration of each respose was 5 times the decay time constant of each case. An alternative comparison of the four Cases in Part (a) is displayed in Fig.2 where all responses are plotted on the same time axis for a duration of 0.5 seconds.

(b) The desired behavioral parameters are $\omega_d = 31.42$ rad/sec (5 Hz) with the following values of damping ratio: Case (i) has $\zeta = 0.1$; Case (ii) has $\zeta = 0.3$; Case (iii) has $\zeta = 0.5$; Case (iv) has $\zeta = 0.7$. From the definitions of ω_d and ζ follow the formulas

$$k = \frac{m\omega_d^2}{1-\zeta^2} \qquad \text{and} \qquad b = \frac{2\zeta m\omega_d}{\sqrt{1-\zeta^2}}$$

The data for Part (b) are assembled in the following Table.

Case	ζ	k [N/m]	b [N/m/sec]	τ [sec]	5τ [sec]	ω_d [r/s]	$\zeta\omega_o$
(i)	0.1	2262	14.32	0.3167	1.584	31.42	0.3167
(ii)	0.3	2460	44.02	0.1012	0.5060	31.42	0.1012
(iii)	0.5	2985	82.28	0.05513	0.2757	31.42	0.0551
(iv)	0.7	4390	139.7	0.03247	0.1624	31.24	0.0325

In Part (b) all four Cases are handled by the MATLAB script 'MassSprgDmpr1.m'. The plots for durations of 5 times the decay time constant are displayed in Fig. 3.

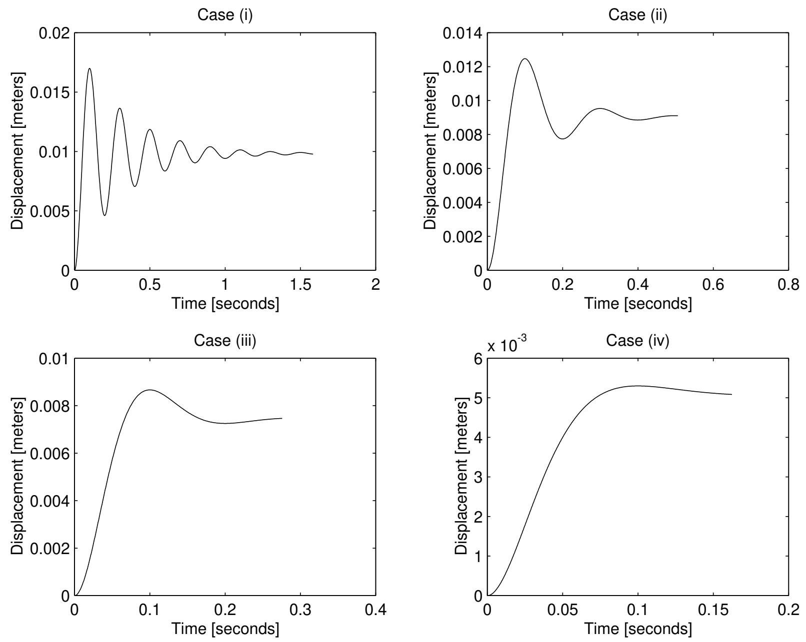


Figure 3: Part (b)

An alternative plot in which all four cases are plotted against the same time scale for a duration of 0.5 seconds is displayed in Fig.4.

(c) The desired behavioral parameters are $\tau = 0.10$ seconds with (i) $\zeta = 0.3$, (ii) $\zeta = 0.5$, (iii) $\zeta = 0.7$, and (iv) $\zeta = 0.9$. From the definitions of τ and ζ

$$\tau = \frac{1}{\zeta\omega_o} \qquad 2\zeta\omega_o = \frac{b}{m}$$

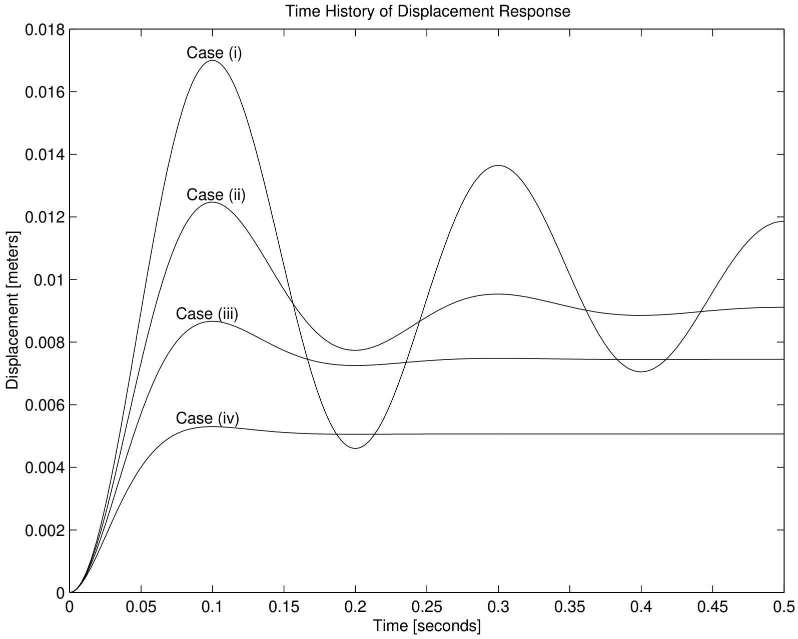


Figure 4: Part (b)

follow the formulas

$$k = \frac{m}{\zeta^2 \tau^2} \qquad \text{and} \qquad b = \frac{2m}{\tau}$$

The data for Part (c) are assembled in the following Table

Case	ζ	k [N/m]	b [N/m/sec]	τ [sec]	5τ [sec]	ω_d [r/s]	$\zeta\omega_o$
(i)	0.3	2520	45.36	0.1	0.5	31.80	10
(ii)	0.5	907.2	45.36	0.1	0.5	17.32	10
(iii)	0.7	462.9	45.36	0.1	0.5	10.20	10
(iv)	0.9	280.0	45.36	0.1	0,5	4.843	10

In Part (c) all four cases are handled by the MATLAB script 'MassSprgDmpr1.m'. The responses for durations of 5 times the decay time constant are plotted against the same time axis in Fig. 5.

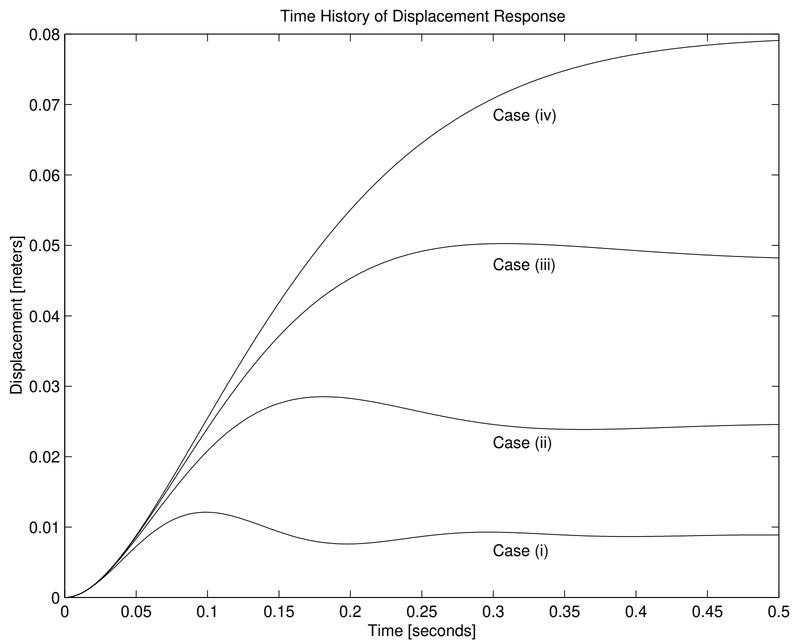


Figure 5: Part (c)

2. The eigenvalues λ for the cases handled by 'MassSprgDmpr1.m' ($0 < \zeta < 1$) are complex conjugates of the form

$$\lambda = -\zeta\omega_o + i\omega_d \qquad \text{and} \qquad \lambda = -\zeta\omega_o - i\omega_d$$

The eigenvalues for the case handled by 'MassSprgDmpr2.m' ($\zeta = 1$) are a pair of repeated real roots: $\lambda = -\omega_o$ and $\lambda = -\omega_o$. The eigenvalues for the case handled by 'MassSprgDmpr3.m' ($1 < \zeta$) are $\lambda_1 = \omega_o(-\zeta + \sqrt{\zeta^2 - 1})$ and $\lambda_2 = \omega_o(-\zeta - \sqrt{\zeta^2 - 1})$. The values of $\zeta\omega_o$ and ω_d for the various cases are listed in the Tables in Problem 1, above. The eigenvalues in the upper half of the complex plane for Part (a) are plotted in Fig. 6. The complex roots for Cases (i) and (ii) have mirror image roots in the lower half-plane.

The eigenvalues in the upper half of the complex plane for Part (b) are plotted in Fig. 7, and the eigenvalues in the upper half of the complex plane for Part (c) are plotted in Fig. 8.

It is instructive to compare the responses for Part (a) in Fig. 2 with the location of the eigenvalues for Part (a) in Fig.6. Likewise for Fig. 4 and Fig.7, for Part (b), and for Fig.5 and Fig.8 for Part (c).

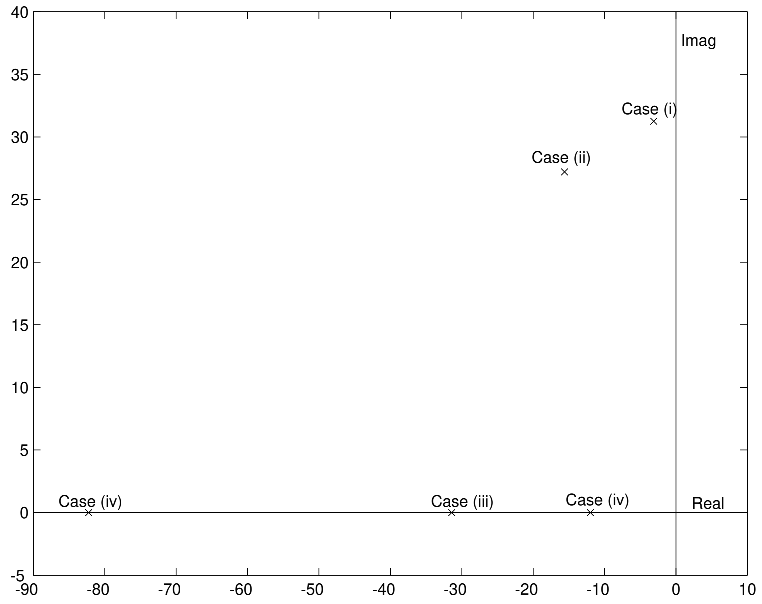


Figure 6: Eigenvalues for Part (a)

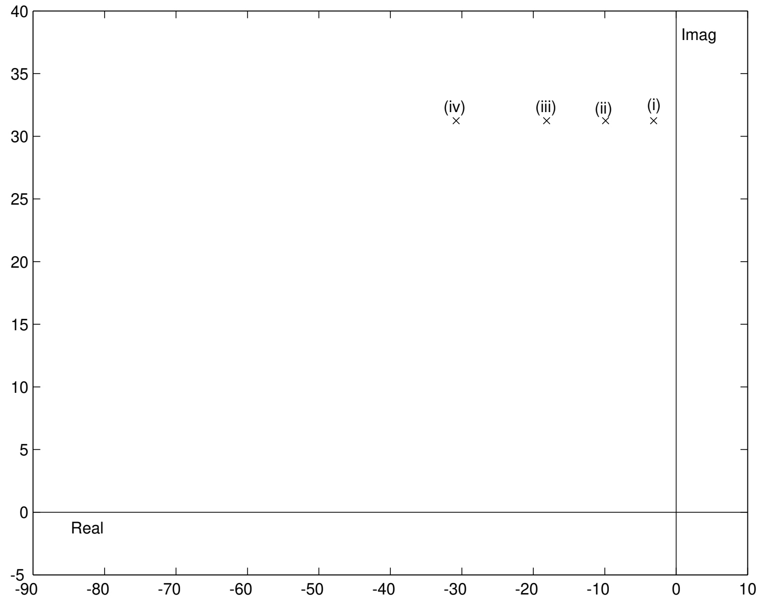


Figure 7: Eigenvalues for Part (b)

3. Here the system of Problem 1 is excited, not by a suddenly applied force, but by the impact of a rubber ball at $t = 0$, which produces an initial velocity $v(0) = v_o$.

(a) The initial velocity is obtained by applying conservation of linear momentum

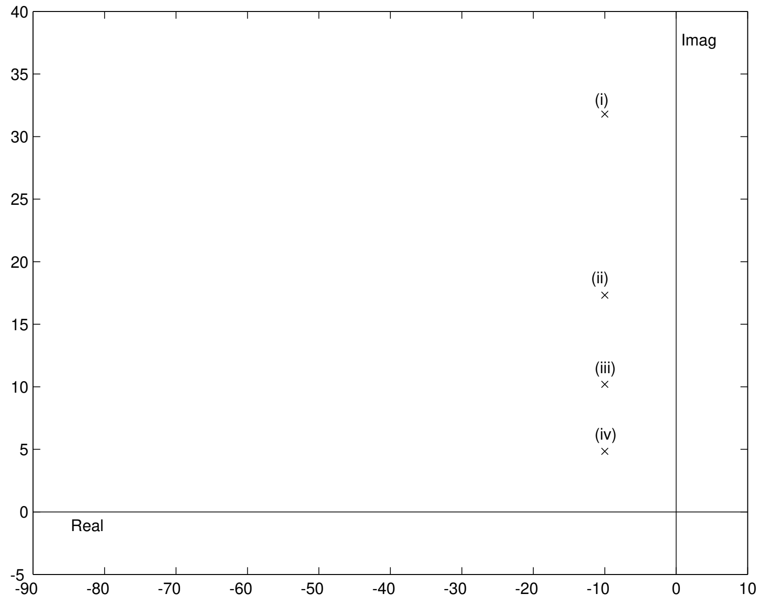


Figure 8: Eigenvalues for Part (b)

to the impact.

$$\begin{aligned}\text{Downward momentum before impact} &= (2\text{pounds})(10 \text{ ft/sec})\\ \text{Downward momentum after impact} &= (5\text{pounds})(v_o) + (2\text{pounds})(\text{-3 ft/sec})\end{aligned}$$

Equating these, yields $v_o = 5.2$ ft/sec $= 1.5850$ m/s (for convenience, it has been assumed that the positive direction for y_k and $v = dy_k/dt$ is downwards).

(b) A damped natural frequency of 5 Hz with $\zeta = 0.3$ is obtained with the parameters found for Case (ii) of Part (b) in Problem 1: $k = 2460$ N/m, and $b = 44.02$ N/m/s.

(c) When these parameters, plus the conditions $f_a = 0$, $y_k(0) = 0$, and $v(0) = 1.5850$ m/s, are input to the MATLAB script 'MassSprgDmpr1.m' the response history labeled "Response to impact of ball" in Fig. 9 is obtained.

(d) In Fig.9 the response to the impact of the ball is compared to the response of the same system to a step-function force of 5 pounds (Case (ii) of Part (b) in Problem 1). In both cases the motion starts from the original equilibrium position of the system. In both cases the mass oscillates for a few cycles and comes to rest. The damped natural frequency and damping ratio is the same in both oscillations. The final rest position is the same as the original equilibrium position in the ball-impact case, while the final rest position in the step-function force case is the new equilirium position under a constant load of 5 pounds. The initial velocity in the step-function force case is zero (displacement response starts with zero slope), while the initial velocity in the ball-impact case is large

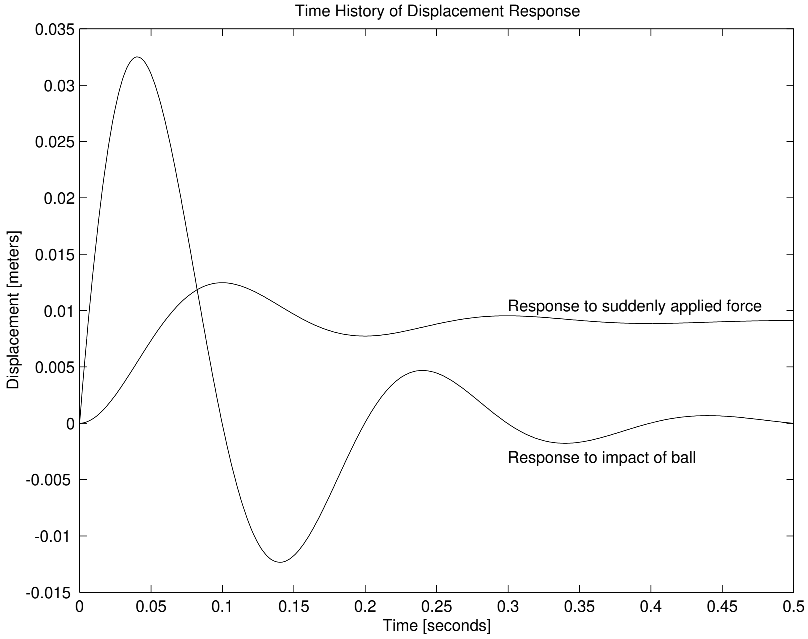


Figure 9: Comparison of responses

due to the impact (displacement response starts with a steep slope). For the particular magnitudes of the excitations given the oscillation in the ball-impact case is more intense than the oscillation due to the suddenly applied force.

4. The basic assumption here is that the back of the truck can be modelled as a plate of mass M supported by an equivalent spring of stiffness k and damping parameter b, onto which hops Uncle Massive, of mass m, which applies a step-function of force to the system, while changing the total mass of the system to $M + m$. Under these assumptions the theory developed in class can be used to estimate the effective mass M of the back of the truck from the observed frequency of oscillation and static deflection.

(a) The statement that the oscillation contained more than five clearly defined cycles is a tip-off that the system is lightly damped, which suggests that damping might be neglected in making a preliminary estimate. The static deflection Δ under a load of $w = mg$ tells us that the equivalent stiffness k can be estimated from $w = k\Delta$, or

$$k = \frac{w}{\Delta} = \frac{250 \text{ pounds}}{1.0 \text{ inch}} = 250 \text{ pounds/inch}$$

The observed oscillation frequency of 2 Hz can be taken as the undamped natural frequency, $\omega_o/2\pi$, of the system with mass $M + m$ and stiffness k. The truck mass M can be estimated by using the relation

$$\omega_o^2 = \frac{k}{M + m}$$

in combination with the preceding expression for k to obtain

$$M = \frac{mg}{\Delta\omega_o^2} - m \tag{1}$$

In the British system of units, mass is subordinate to force, and M is replaced by W/g where W is the equivalent weight of the back of the truck. To use British units, Eq.(1) is multiplied through by $g = 386$ in/sec^2 to get

$$W = \frac{wg}{\Delta\omega_o^2} - w = \frac{(250)(386)}{(1)(4\pi)^2} - 250 = 361 \text{ pounds}$$

Alternatively the given data can be converted to SI units ($\Delta = 0.0254$ m, and $m = 113.4$ kg),and Eq.(1) used to obtain $M = 163.8$ kg whose weight, $Mg = (163.8)(9.81) = 1607$ Newtons, is eqivalent to the British weight of 361 pounds.

(b) It is assumed that the truck oscillation is a primarily vertical motion of the effective mass of the rear end of the truck plus Uncle Massive on an effective vertical spring (the rear tires plus suspension). It is also assumed that the observed *damped* natural frequency is a good approximation for the *undamped* natural frequency required in the formulas above.