2.003 Fall 1999 Solution of Homework Assignment 4

1. Due to the application of a 1.0 Newton step-force, the system oscillates at its damped natural frequency ω_d about the new equilibrium position $y_k = \Delta$. From the given Displacement Response plot, the equilibrium offset Δ can be estimated to be $\Delta = 4.4 \text{ x } 10^{-4}$ meters. The damped natural frequency can be estimated by counting the cycles, 10, in 2.0 seconds, which gives a damped period $T_d = 0.2$ seconds, and $\omega_d = 2\pi/T_d = 31.4$ rad/sec. The logarithmic decrement LDR can be estimated by measuring the ratios of successive peak amplitudes. This is generally a difficult measurement to make accurately. One way to increase the accuracy is to make a large number of measurements based on different pairs of successive peaks, and average the results. Draw a line across the plot at $y = \Delta$ and measure the amplitudes of several peaks. Since only the ratio of successive peaks will be used, the measurements can be in terms of any convenient length unit. The measurements below are in millimeters.

Station	*Amplitude*	*Ratio*
0	6.22	–
1	4.86	0.781
2	4.00	0.823
3	3.08	0.770
4	2.56	0.831
5	1.94	0.758
6	1.68	0.866

The average of these first six ratios is 0.805, so the estimate for the LDR is $ln\, 0.805 = -0.217$.

(a) The estimated stiffness is

$$k = \frac{f_a}{\Delta} = \frac{1.0}{4.4 \times 10^{-4}} = 2270 \text{ Newtons/meter}$$

(b) To estimate the mass m, when the stiffness k is known, it is necessary to know the *undamped* natural frequency ω_o. At this point we know only the *damped* natural frequency ω_d. We can postpone estimation of the mass m until the damping ratio ζ is estimated in (c) below, or we can assume, since more than ten cycles of oscillation are visible, that the damping is sufficiently light to permit us to make the approximation that $\omega_o \approx \omega_d$, in which case

$$m \approx \frac{k}{\omega_d^2} = \frac{2270}{(31.4)^2} = 2.30 \text{ kg}$$

(c) To estimate the damping parameter b, we need to know the damping ratio ζ. The damping ratio follows from the log decrement ratio, LDR, according to the formula

$$\zeta^2 = \frac{LDR^2}{\pi^2 + LDR^2}$$

given in the Notes accompanying Lecture 6. Insertion of the estimation LDR $= -0.217$ yields $\zeta \approx 0.069$. (Note: With this value of ζ the undamped natural frequency is estimated as follows

$$\omega_o = \frac{\omega_d}{\sqrt{1-\zeta^2}} \approx \frac{31.4}{\sqrt{1-(0.069)^2}} = 31.5 \text{ rad/sec}$$

With this estimate for ω_o the mass m, previously estimated in (b) above at 2.30 kg, would now be estimated to be 2.29 kg. The damping parameter b is then estimated as

$$b = 2\zeta\omega_o m \approx 2(0.069)(31.5)(2.29) = 9.95 \text{ kg/sec, or } 9.95 \text{ N/m/sec}$$

2. With two plate-on-springs units face-to-face, the effective stiffness of the combined system is $k = 1000$ N/m, and the effective damping parameter is $b = 10$ N/m/sec. For free vertical motion of the mass $m = 2.0$ kg, the displacement y_k from the equilibrium position of the model satisfies the equation

$$m\frac{d^2y_k}{dt^2} + b\frac{dy_k}{dt} + ky_k = 0$$

(a) The undamped natural frequency ω_o for the model is

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{2}} = 22.4 \text{ rad/sec, or } f_o = \frac{\omega_o}{2\pi} = 3.56 \text{ Hz}$$

(b) The damping ratio ζ for the model is

$$\zeta = \frac{b}{2\omega_o m} = \frac{10}{2(22.4)(2)} = 0.1116$$

(c) The decay time constant for the model is

$$\tau = \frac{1}{\zeta\omega_o} = \frac{1}{(0.1116)(22.4)} = 0.400 sec$$

3. Initially the bungee jumper free-falls under the influence of gravity and air resistance, then when the slack is taken up in the elastic cord, the cord exerts an increasing retarding force which reduces her velocity to zero at level A, after which she bobs up and down with decreasing amplitude until she comes to rest at the equilibrium position at level B. For the purposes of a preliminary estimate of the conditions up to instant at which level A is reached, neglect the effects of air resistance and damping in the cord. Under this assumption energy is conserved. In the first jump the equilibrium position B is 20 feet below the point where the cord begins to stretch. The stiffness of the cord is then estimated as

$$k = \frac{W}{\Delta} = \frac{150}{20} = 7.5 \text{ pounds/foot}$$

The undamped natural frequency of the mass-spring system consisting of the jumper and cord is

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{W/g}} = \sqrt{\frac{7.5}{150/32.2}} = 1.269 \text{ rad/sec}$$

Let the elevation of the upper attachment point of the cord be denoted by h_o, the elevation of the point where the slack is taken up and the cord begins to stretch be enoted by h_1, the elevation of the equilibrium point B be denoted by h_B, and the elevation of the point A where the jumper's velocity first vanishes be denoted by h_A.

(a) In the first jump, $h_o - h_1 = 100$ feet, and $h_o - h_B = 120$ feet. The level A can be located by equating the potential energy lost in the fall to the elastic energy in the cord.

$$W(h_o - h_A) = k\Delta(h_o - h_A) = \frac{1}{2}k(h1 - h_A)^2 = \frac{1}{2}k[(h_o - h_A) - 100]^2$$

This reduces to a quadratic equation for $(h_o - h_A)$ whose solutions are

$$(h_o - h_A) = 120 \pm 66.3 \text{ feet}$$

The physically significant root is $(h_o - h_A) = 186.3$ feet. The low point A is 66.3 feet below the equilibrium level B, or 186.3 feet below the upper attachment point.

(b) The maximum downward acceleration is 32.2 feet/sec^2 during the initial free fall.

(c) The maximum upward acceleration occurs at point A and is

$$a_{max} = (\text{Max displacement from equilibrium})\omega_o^2 = (66.3)(1.269)^2 = 106.8\text{ft/sec}^2$$

which is 3.32 times the acceleration of gravity.

(d) The primary assumption made is that dissipation of energy has been neglected. The cord has been assumed to behave like a linear spring.

(e) In the second jump the slack length is doubled which cuts the stiffness in half, and doubles the distance Δ to 40 feet. The natural frequency is reduced to 0.897 rad/sec. The location of the new low point A' is obtained by solving a similar quadratic equation with the new values of $h_0 - h_1 = 200$ feet, $h_o - h_{B'} = 240$ feet, and $\Delta = 40$ feet. The result is

$$(h_o - h_{A'}) = 240 \pm 132.7 \text{ feet}$$

The physically significant root is $(h_o - h_A) = 373$ feet. The low point A is 132.7 feet below the equilibrium level B', or 373 feet below the upper attachment point.

(f) The level of the equilibrium point B' is 240 feet below the attachment point.

(g) The maximum downward acceleration is still 32,2 feet/sec

(h) The maximum upward acceleration is

$$a_{max} = (\text{Max displacement from equilibrium})\omega_o^2 = (132.7)(0.897)^2 = 106.8\text{ft/sec}^2$$

Although the jump involves a longer free fall and a greater extension of the cord the maximum acceleration does not change!

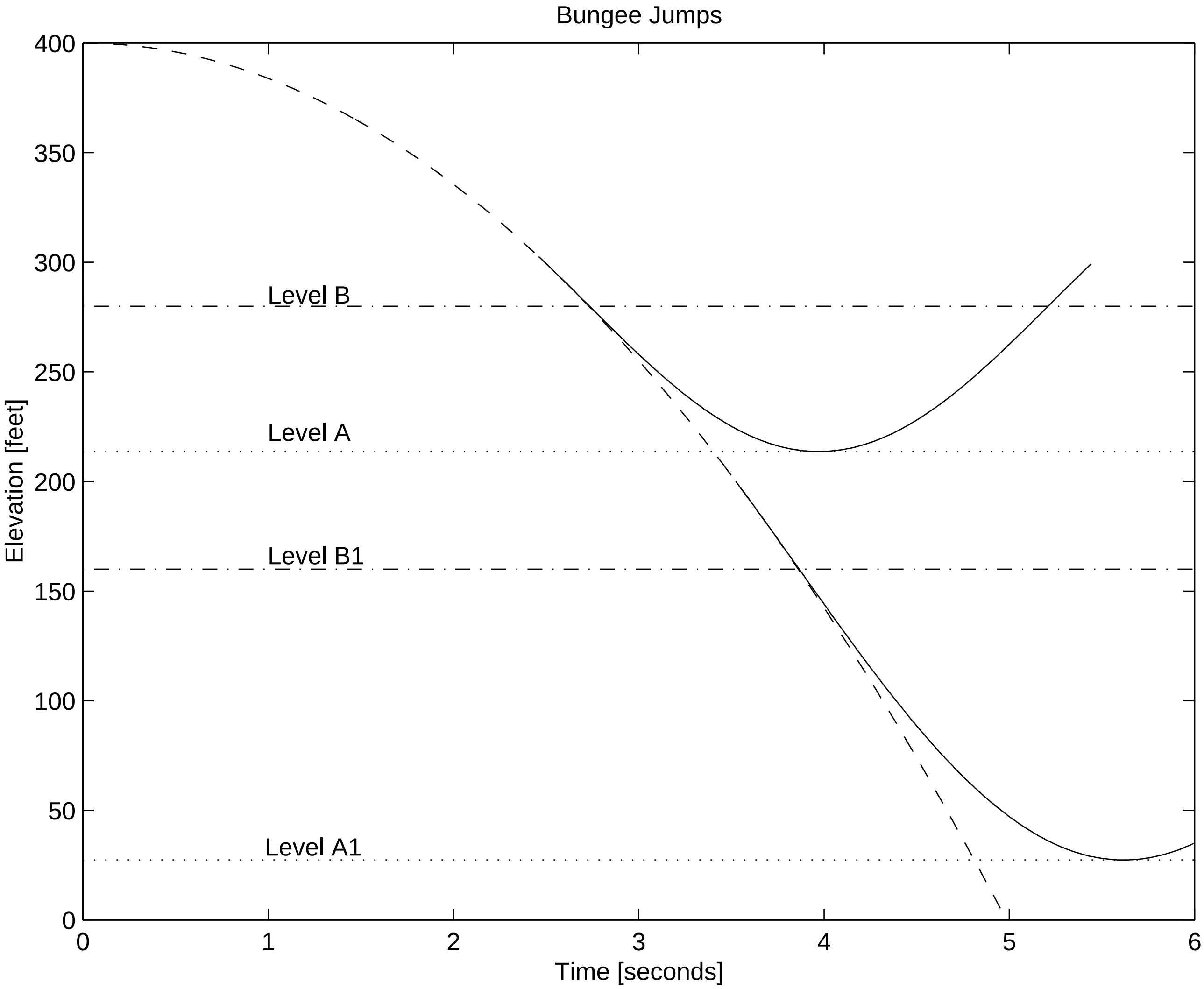

Figure 1: Time Histories of Bungee Jumps

(i) The time histories of the two jumps are shown in Fig.1. For plotting purposes the elevation of the upper attachment point was (arbitrarily) assigned the value 400 feet. The trajectory of free fall is indicated by the dashed-line parabola.

With 100 feet of slack the jumper's trajectory departs from the free-fall parabola at $h = 300$ ft and begins to oscillate about the final equilibrium level B. With no damping the maximum excursion (and maximum acceleration) is at Level A. With 200 feet of slack the jumper's trajectory departs from the free-fall parabola at $h = 200$ ft and begins to oscillate about the final equilibrium level B1. With no damping the maximum excursion (and maximum acceleration) is at Level A1.

4. Let x be the displacement of the engine with respect to the stationary crate. The effective stiffness of the two end elements is $2k$, and the effective damping parameter is $2b$.

(a) The equation of motion for the engine is

$$m\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + 2kx = 0$$

(b) The engine weight is $W = 500$ pounds. Its mass is $m = W/g$. The values of k and b can be deduced from the given values of damped natural frequency $\omega_d = 2\pi$ and damping ratio $\zeta = 0.707$ from the equations

$$\omega_o^2 = \frac{\omega_d^2}{1-\zeta^2} = \frac{2k}{W/g} \qquad \text{and} \qquad 2b = 2\zeta\omega_o W/g$$

The undamped natural frequency is

$$\omega_o = \frac{2\pi}{\sqrt{1-(0,707)^2}} = 8.88 \text{ rad/sec}$$

and

$$k = \frac{W\omega_o^2}{2g} = \frac{(500)(8.88)^2}{2(386)} = 51.2 \text{ pounds/inch}$$

The damping parameter is

$$b = \zeta\omega_o\frac{W}{g} = (0.707)(8.88)\frac{500}{386} = 8.13 \text{ pounds/in/sec}$$