2.003 Fall 1999 Solution of Homework Assignment 5

1. (30 Points) Fig.1 shows an idealized model of the engine of mass $m = W/g$ with shock-absorbing packaging, represented by the springs with stiffness k and damping parameter b, in a crate which is fixed to a truck. The velocity of the truck, with respect to an inertial reference frame, is v_{truck}, and the velocity of the engine, with respect to an inertial reference frame, is v_{engine}. The relative displacement of the engine with respect to the crate is x_{rel}. The origin for x_{rel} is taken to be the equilibrium position of the engine in the motionless crate. In problems like this, where relative motion is involved, it is important to remember that Newton's law only applies to motions with respect to an *inertial* reference frame

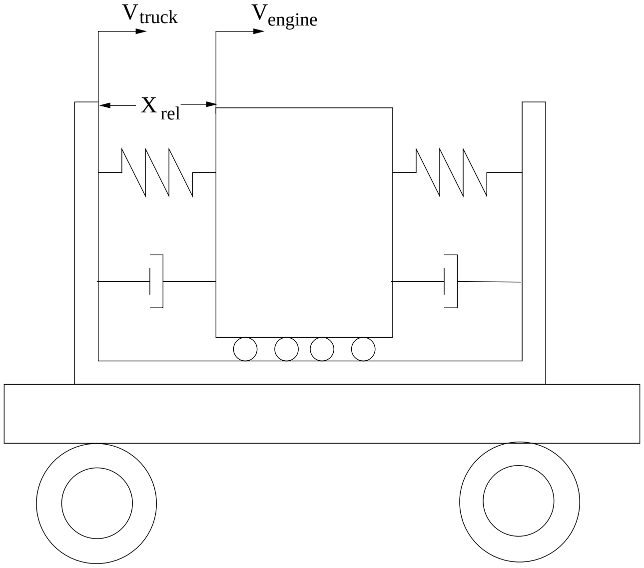

Figure 1: Model of Engine in Crate on Moving Truck

(a) The *geometric compatibility* conditions are

$$\frac{dx_{rel}}{dt} = v_{rel} \qquad \text{and} \qquad v_{engine} = v_{truck} + v_{rel}$$

The *constitutive equations* are

$$f_m = m\frac{dv_{engine}}{dt}, \qquad f_{spring} = kx_{rel}, \qquad \text{and} \qquad f_{damping} = bv_{rel}$$

and the *force balance* condition is

$$f_m = -2f_{spring} - 2f_{damping}$$

The six preceding equations constitute a mathematical model for the longitudinal motion of the engine with respect to the crate.

(b) A single differential equation in terms of x_{rel} can be derived by inserting the spring force $f_{spring} = kx_{rel}$ and the damping force $f_{damping} = bdx_{rel}/dt$ in the force-balance equation, along with f_m expressed as $f_m = m(dv_{truck}/dt + d^2x_{rel}/dt^2)$. The result is

$$m\frac{d^2x_{rel}}{dt^2} + 2b\frac{dx_{rel}}{dt} + 2kx_{rel} = -m\frac{dv_{truck}}{dt}$$

The form of this equation is similar to that for the equation for the displacement of the steel plate on springs. Here the unknown response is a *relative* displacement and the driving force is the negative of the force that would be required to give the engine the same acceleration as the truck. A single differential equation for the relative *velocity* of the engine with respect to the crate is obtained by differentiating every term in the preceding equation.

$$m\frac{d^2v_{rel}}{dt^2} + 2b\frac{dv_{rel}}{dt} + 2kv_{rel} = -m\frac{d^2v_{truck}}{dt^2}$$

(c) A state-determined system for the state variables x_{rel} and v_{rel} can be obtained by expressing the forces in the force-balance equation of (a) in terms of both x_{rel} and v_{rel}. The two first-order differential equations for the state-determined system are

$$\begin{aligned}\frac{dx_{rel}}{dt} &= v_{rel}\\ \frac{dv_{rel}}{dt} &= -2\frac{k}{m}x_{rel} - 2\frac{b}{m}v_{rel} - \frac{dv_{truck}}{dt}\end{aligned}$$

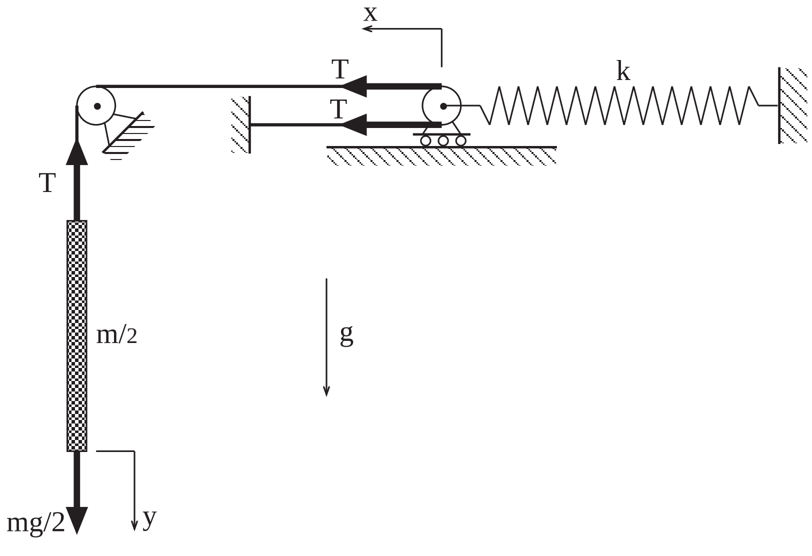

Figure 2: Garage Door support System

2. (40 Points: 20 point for part a, 10 points for part b,c) To model the pulley system, we assume that the inertia of the pulleys and cable is negligible in comparison with the

mass of the garage door. Furthermore we assume that the cable is flexible (it bends easily) but stretches very little in comparison with the extension of the spring. Let x be the horizontal displacement of the pulley attached to the end of the spring, and let y be the vertical displacement of the door, both measured from the equilibrium position of the door hanging freely under the influence of gravity. In the equilibrium position, the tension T_o in the cables must equal the weight mg of the door. The total force in the spring is $2T_o$ which has produced an initial extension Δ of the springs, where $\Delta = 2T_o/k = 2mg/k$.

In a small oscillation about the equilibrium position, the displacements $x(t)$ and $y(t)$ oscillate about zero and the tension in the cable oscillates about the initial tension $T_o = mg$. When the cable remains taut, the displacements x and y are linked by the geometric compatibility requirement $y = 2x$. This is the crucial insight for this system, and it is not immediately obvious to all. One way to see it is to imagine that the door and the cables are temporarily frozen in position while the pulley attached to the spring is displaced a distance x to the left. An empty loop of cable appears, to the right of the pulley, with a straight length x in the upper cable and a straight length x in the lower cable. Now with the pulley fixed, gradually lower the door to take up the slack in the empty loop. It will be necessary to lower the door a distance $y = 2x$ to make the cable taut again.

(a) To formulate a model to analyze the door oscillations we need to assemble the geometric compatibility requirements, the constitutive equations, and the force balance requirements. The *geometric compatibility* requirement is

$$y = 2x$$

The *constitutive equations* for the door and the spring are

$$f_m = \frac{m}{2}\frac{d^2y}{dt^2} \qquad \text{and} \qquad 2T = k(\Delta + x) = 2mg + kx$$

and the *force balance* requirement is

$$f_m = mg - T$$

These relations constitute a mathematical model of the garage door system. They can be organized into a single differential equation for either x or y by inserting the constitutive equations for f_m and T in the force balance equation and using the compatibility requirement to eliminate x in favor of y, or *vice versa.* The result is either

$$m\frac{d^2y}{dt^2} + \frac{k}{2}y = 0 \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + \frac{k}{2}x = 0$$

Note that because the origins for the displacements x and y were established at the equilibrium position, the gravity force has cancelled out of the equations of motion.

(b) The undamped natural frequency ω_o for the preceding equations is given by

$$\omega_o^2 = \frac{k}{2m} = \frac{kg}{2W}$$

where W is the weight of the door and $g = 386$ in/sec^2 is the acceleration of gravity. The frequency of oscillation f_o in Hz is

$$f_o = \frac{\omega_o}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{(5)(386)}{2(200)}} = 0.350 \text{ Hz}$$

(c) It is assumed that the door only moves vertically in the small oscillation. The inertia of the pulleys, cable, and spring is neglected. The elasticity of the cable in tension is neglected, and it is assumed that the tension T is uniform throughout the length of the cable.

3. (30 Points) To obtain a state-determined system, introduce the vertical velocity v of the plate. The *geometrical compatibility* requirement is

$$\frac{dy}{dt} = v$$

The *constitutive* equations are

$$f_{springs} = ky \qquad \text{and} \qquad f_{damping} = bv$$

and the *force balance* requirement is

$$f_m = -f_{springs} - f_{damping}$$

The dynamic state equations are two first-order differential equations for y and v

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= \frac{1}{m}f(t) - \frac{k}{m}y - \frac{b}{m}v \end{aligned}$$

(a) The deflection Δ under the weight mg is $\Delta = mg/k$ so the mass m is given by

$$m = \frac{k\Delta}{g} = \frac{(3000)(0.007)}{9.81} = 2.14 \text{ kg}$$

The undamped natural frequency is $\omega_o = \sqrt{k/m} = \sqrt{3000/2.14} = 37.4$ rad/sec, and the damping parameter is

$$b = 2\zeta\omega_o m = 2(0.5)(37.4)(2.14) = 80.0 \text{ N/m/s}$$

(b) The dynamic state equations can be rewritten in matrix form

$$\frac{d}{dt}\left\{ \begin{array}{c} y \\ v \end{array} \right\} = \left[\begin{array}{cc} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{array} \right] \left\{ \begin{array}{c} y \\ v \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ \dfrac{f_a}{m} \end{array} \right\}$$

or

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{u}$$

where

$$\mathbf{x} = \left\{ \begin{array}{c} y \\ v \end{array} \right\} \qquad \mathbf{A} = \left[\begin{array}{cc} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{array} \right] \qquad \mathbf{u} = \left\{ \begin{array}{c} 0 \\ \dfrac{f_a}{m} \end{array} \right\}$$

where f_a is the magnitude of the constant applied force. These equations can be integrated, from prescribed initial values of y and v at $t = 0$ by the MATLAB command "ode45". A possible script for doing this is displayed below.

```
% 'POS.m' A MATLAB script for Problem 3 in 2.003 Assignment 5. Produces plots of
%   (i) position vs. time
%  (ii) velocity vs. time
% (iii) velocity vs. position
% for the response of a steel plate on springs, with mass m, stiffness k,
% and damping parameter b, when the plate starts from initial conditions
% y = y0 and v = v0 under the action of a suddenly applied force fa at t = 0.

% 'POS.m' This script
clear variables
global m k b fa
% Input parameters
m = input('Enter the mass "m" in kilograms ');
k = input('Enter the stiffness "k" in Newtons/meter ' );
b = input('Enter the damping constant "b" in kilograms/sec ');
fa= input('Enter the magnitude "fa" of the suddenly applied force in Newtons ');
% Input initial conditions.
y0= input('Enter the initial displacement from equilibrium, in meters ');
v0= input('Enter the initial velocity, in meters/second ');
tspan = input('Enter the duration "T" of the desired time history, in seconds ')
X0 = [ y0 ; v0 ];
% Integrate equations of motion
[t,X] = ode45('pl_on_spr', tspan, X0);
% Plot results
plot(t,X(:,1)), title('Time History of Displacement'),
xlabel('Time  [Seconds]'), ylabel('Displacement  [meters]'), pause
plot(t,X(:,2)), title('Time History of Velocity'),
```

```
xlabel('Time  [Seconds]'), ylabel('Velcity  [meters/second]'), pause
plot(X(:,1), X(:,2)), title('Velocity  vs. Displacement'),
xlabel('Displacement  [meters]'), ylabel('Velcity  [meters/second]')
```

The formula 'pl_on_spr' which is called by 'POS.m' follows.

```
% 'pl_on_spr.m'  Provides equation of motion for plate on springs
% to be integrated by script POS.m

function Xdot = pl_on_spr(t,X)
global m k b fa
Xdot = [ 0  1 ; -k/m -b/m ]*X + [ 0 ; fa/m ];
```

(c) When the scripts above are run with the following inputs

$m = 2.14$ kg

$k = 3000$ N/m

$b = 80.0$ N/m/s

$f_a = mg = (2.14)(9.81) = 21.0$ Newtons

the following three plots are obtained for a time interval of T = 0.35 seconds. There is very little activity in interval between $t = 0.35$ and $t = 2.0$ seconds.

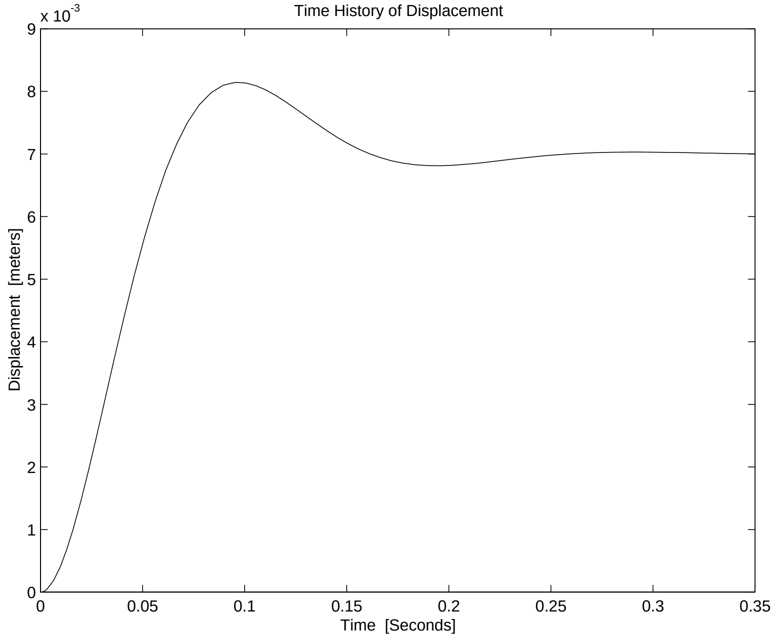

Figure 3: Plot of Position vs.Time

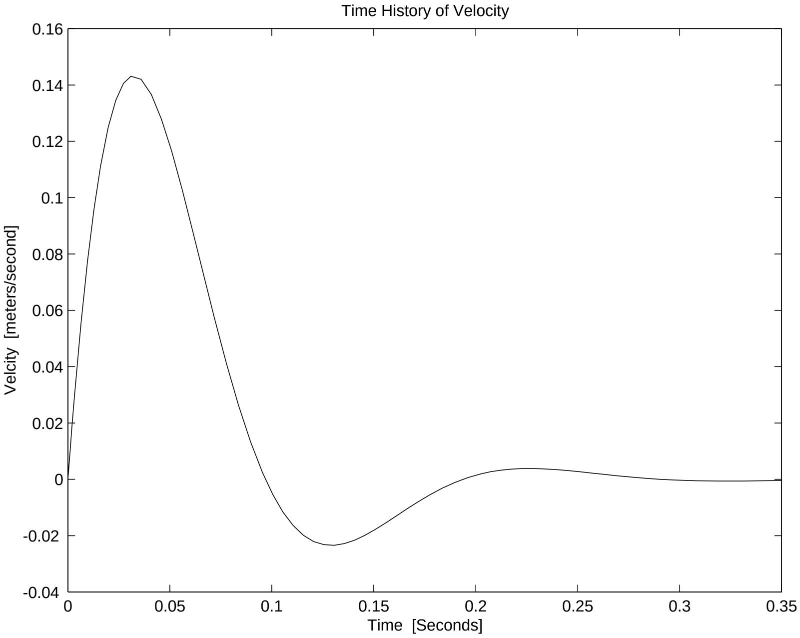

Figure 4: Plot of Velocity vs.Time

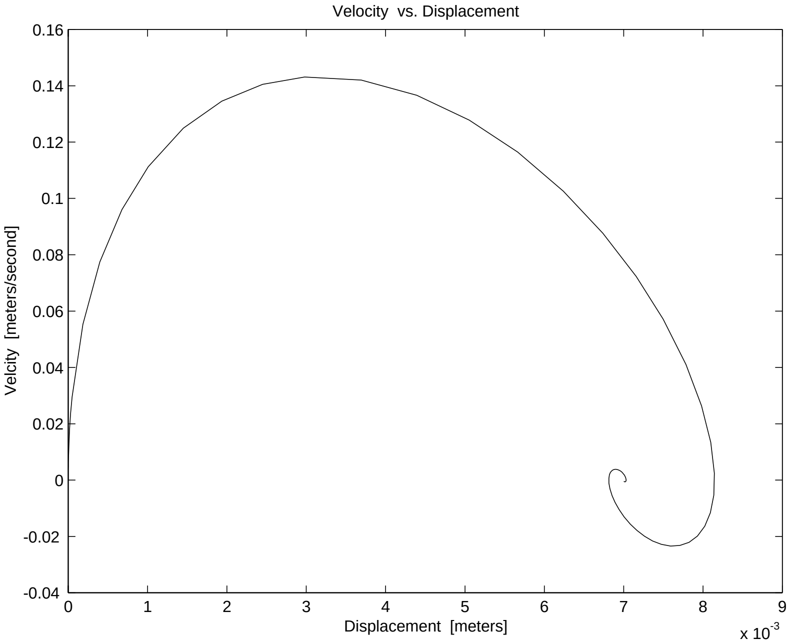

Figure 5: Plot of Velocity vs.Position