2.003 Fall 1999 Solution of Homework Assignment 6

1. (36 Points, 4 points each part) Assume hovering ballon has mass m and is subject to a buoyancy force B which tends to lift the balloon. Consider only vertical motion of the balloon. Let the length of the string between the balloon and the floor be y. Assume the string has mass/unit length ρ so that a string mass of ρy must move whenever the balloon moves. Both the balloon mass and the string mass experience gravitational attraction pulling them down. Assume that there is no tension in the string at the point where the string meets the floor. Finally assume there are damping mechanisms (air drag and losses associated with coiling and uncoiling the string on the floor) which can be approximately modeled by a linear dampimg force $f_d = bdy/dt$ opposing the velocity dy/dyt. These forces are displayed in the free body diagram in Fig.1.

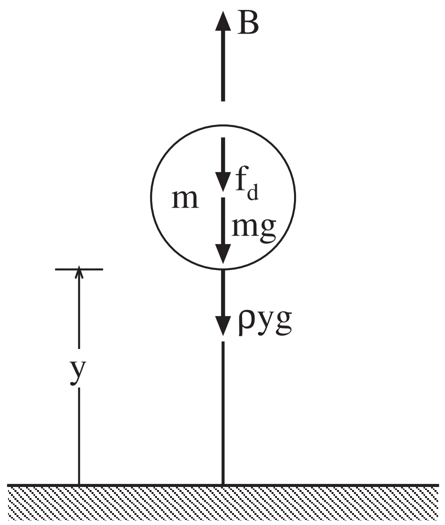

Figure 1: Free-Body Diagram of Hovering Balloon

To construct a model we consider (i) geometric compatibility requirements, (ii) constitutive equations, and (iii) force-balance requirements.

(i) The displacement of the balloon is $y(t)$. The velocity and acceleration are

$$v = \frac{dy}{dt} \qquad a = \frac{dv}{dt} = \frac{d^2y}{dt^2}$$

(ii) The consitutive equations are

$$f_m = \frac{d}{dt}[(m + \rho y)v] \qquad f_d = bv \qquad f_{gravity} = (m + \rho y)g$$

(iii) The force-balance requirement is

$$f_m = B - f_{gravity} - f_d$$

An equation of motion for the model is obtained by substituting the constitutive equations into the force-balance requirement to get

$$\frac{d}{dt}[(m + \rho y)v] + b\frac{dy}{dt} + (m + \rho y)g = B \tag{1}$$

(a) In a steady state y has a fixed value y_o and all time derivatives. When we set the time derivatives equal to zero in (1) we find that the equation is satisfied if

$$y = y_o = \frac{B - mg}{\rho g}$$

The variable mass term in (1) makes the equation nonlinear. It can be approximated by a linear equation if the varying mass of the string, ρy is replaced by the fixed mass ρy_o of the string in its equilibrium position. This is a reasonable approximation for studying *small* oscillations about the equilibrium position. The dynamics of such oscillations are clarified if we introduce the displacement $z = y - y_o$ of the ballon from its equilibriium position. After inserting $y = y_o + z$ in (1) we find

$$(m + \rho y_o)\frac{dv}{dt} + b\frac{dz}{dt} + \rho g z = B - (m + \rho y_o)g = 0$$

Finally, introducing the total mass $M = m + \rho y_o$ and putting $v = dy/dt = dz/dt$, we obtain

$$M\frac{d^2 z}{dt^2} + b\frac{dz}{dt} + \rho g z = 0 \tag{2}$$

This is the standard form for free oscillation of a linear second-order system.

(b) Physically, when the ballon rises above its equilibrium position it lifts additional string whose weight tends to lower the balloon, and when the balloon descends below its equilibrium position it carries a lighter load of string which tends to drive the balloon upward. The equivalent stiffness in the equation of motion (2) is $k = \rho g$. This is the weight per unit length of the string. This example illustrates the fact that restoring-force elements do not always look like springs.

(c) The second-order equation of motion (2) predicts oscillation if the damping is light enough to produce a critical damping factor ζ smaller than unity.

(d) A free vibration governed by Eq. (2) will eventually decay to zero if there is a positive damping parameter b.

To obtain the behavioral parameters ζ and ω_o we use the equations

$$\zeta^2 = \frac{LDR^2}{\pi^2 + LDR^2} \qquad \text{and} \qquad \omega_o = \frac{\omega_d}{\sqrt{1 - \zeta^2}}, \qquad \text{with} \qquad \omega_d = \frac{2\pi}{T_d}$$

(e) From the given data we see that the ratio of successive peak amplitudes is 0.2. Thus the LDR is $\ln(0.2)$ = -1.609, and the damping ratio is

$$\zeta = \left(\frac{(1.609)^2}{\pi^2 + (1.609)^2}\right)^{1/2} = 0.456$$

(f) From the given data we see that the damped natural period T_d is 3.0 seconds, so that the damped natural frquency is $\omega_d = 2\pi/T_d = 2.094$ rad/sec, and

$$\omega_o = \frac{2.094}{\sqrt{1 - (0.456)^2}} = 2.35 \text{ rad/sec}$$

To obtain the model parameters, we begin with the effective stiffness k which is the weight per unit length of the string. This is given as 0.5 ounces per foot which is 0.03125 pounds/foot in the English system or

$$k = 0.03125 \cdot \frac{4.448}{0.3048} = 0.456 \text{ Newtons/meter}$$

(g) The effective mass M of the balloon plus the equilibrium length of string, is given by

$$M = \frac{k}{\omega_o^2} = \frac{0.456}{(2.35)^2} = 0.0826 \text{ kg}$$

in the SI system. In the English system

$$M = \frac{W}{g} = \frac{W}{32.2} = \frac{0.03125}{(2.35)^2} = 0.00566 \text{ pound sec}^2/\text{ft}$$

from which we find W = 0.1823 pounds or 2.92 ounces. The weight of 3 feet of string is 1.5 ounces so the weight of the balloon is 1.42 ounces. In the SI system, the mass of 3 feet of string is 3(0.3048)(0.456)/9.81 = 0.0425 kg, so the mass of the balloon is m = 0.0826 - 0.0425 =0.0401 kg.

(h) The effective damping coefficient $b = 2\zeta\omega_o M$ is

$$b = 2(0.456)(2.35)(0.0826) = 0.1770 \text{ N/m/s}$$

in SI units, or

$$b = 2(0.456)(2.35)(0.00566) = 0.0121 \text{ pounds/ft/sec}$$

(i) The effective stiffness was previously obtained as 0.456 Newtons/meter in SI units or 0.03125 pounds/foot in English units.

2. (20 Points, 5 points each part) Let the effective combined stiffness of the four unblocked springs be k_u, and let the effective combined stiffness of the four blocked springs be k_b. In both cases the vehicle weight mg is supported by the springs. Since only half of the spring is active in the blocked case the static deflection $\Delta_b = mg/k_b$ in the blocked case will be half the static deflection $\Delta_u = mg/k_u$ in the unblocked case. Since we are given that $\Delta_u - \Delta_b = 2.5$ inches, we deduce that $\Delta_u = 5.0$ inches, and $\Delta_b = 2.5$ inches.

(a) The effective stiffnesses are

$$k_u = \frac{2500}{5} = 500 \text{ pounds/inch} \qquad \text{and} \qquad k_b = \frac{2500}{2.5} = 1000 \text{ pounds/inch}$$

in British units. In SI units

$$k_u = 500\frac{4.448)}{0.0254} = 87,560 \text{ Newton/meter} \quad \text{and} \quad k_b = 1000\frac{4.448)}{0.0254} = 175,100 \text{ Newton/meter}$$

(b) The undamped natural frequency ω_o is given by $\omega_o^2 = k/m$ where m is the mass of the vehicle. In SI units $m = 2500(0.4536) = 1134$ kg. The natural frequencies are

$$(\omega_o)_u = \sqrt{\frac{87,560}{1134}} = 8.787\text{rad/sec} \qquad \text{and} \qquad (\omega_o)_b = \sqrt{\frac{175,100}{1134}} = 12.437\text{rad/sec}$$

(c) We are given that the unblocked suspension is critically damped ($\zeta_u = 1.0$). The effective damping coefficient is

$$b = 2\zeta_u(\omega_o)_u m = 2(1.0)(8.787)(1134) = 19,930 \text{ Newtons/meter/second}$$

(d) Assuming that the shock absorbers are unchanged by the blocking operation, the damping coefficient remains $b = 19,930$ N/m/s and the new value of ζ for the system with blocked springs is

$$\zeta_b = \frac{b}{2(\omega_o)_b m} = \frac{19,930}{2(12.437)(11340} = 0.7071$$

If y is measured vertically upward from the equilibrium position, and $v = dy/dt$, the constitutive equations are

$$f_m = m\frac{dv}{dt}, \qquad f_d = bv, \qquad \text{and} \qquad f_k = ky$$

with appropriate subscripts u and b for the unblocked and blocked cases, respectively. and the force-balance requirement is

$$f_m = f(t) - f_d - f_k$$

where $f(t)$ is the upward driving force applied to the vehicle. These model requirements can be organized into the following state equations for a state determined system with state variables y and v.

$$\frac{d}{dt}\left\{\begin{array}{c} y \\ \\ v \end{array}\right\} = \left[\begin{array}{cc} 0 & 1 \\ \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{array}\right]\left\{\begin{array}{c} y \\ \\ v \end{array}\right\} + \left\{\begin{array}{c} 0 \\ \\ \dfrac{f(t)}{m} \end{array}\right\}$$

The output equation for the displacement response $y(t)$ is

$$y(t) = \{\ 1 \quad 0\ \}\left\{\begin{array}{c} y \\ \\ v \end{array}\right\}$$

The state equations can be integrated by adapting the scripts 'POS.m' and 'pL_on_spr.m' for the Plate on Springs problem of Assignment 5 to apply to the present problem. The adaptation of 'POS.m' involved inputting the specific values of m, b, f_a, x_o, and v_o and eliminating unecessary plots. The adapted version of 'POS.m' is called 'HW6Pr2.m'. The script for 'HW6Pr2.m' follows:

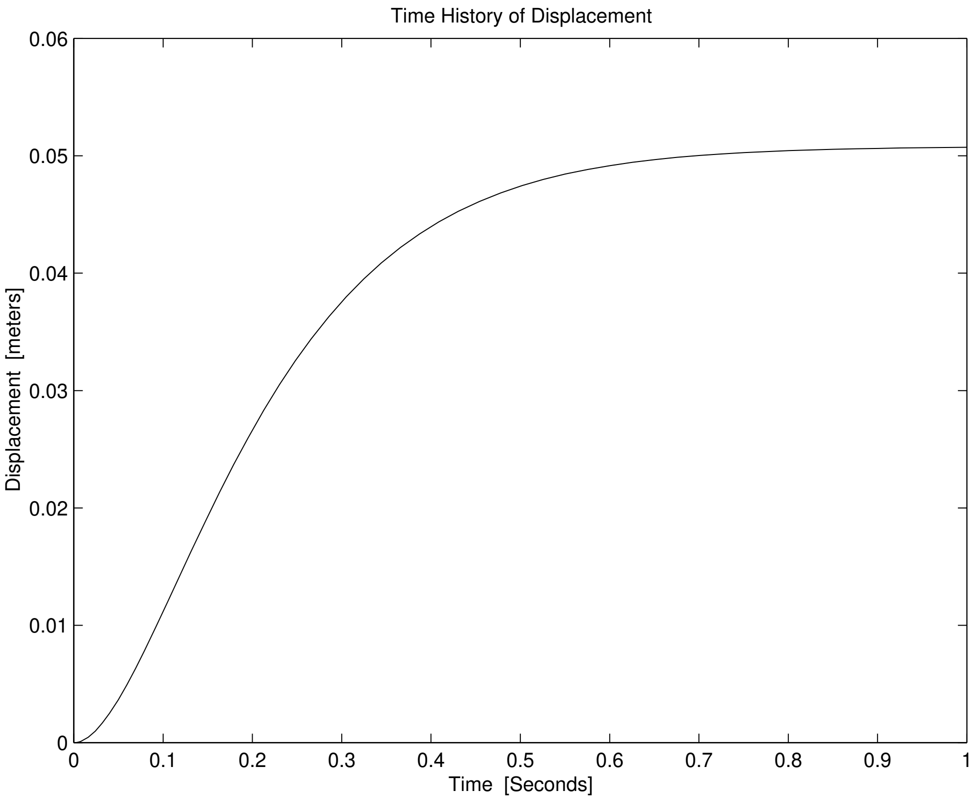

Figure 2: Response of Unblocked Vehicle

```
% HW6Pr2.m MATLAB Script adapted from POS.m which was
% MATLAB script for Problem 3 in 2.003 Assignment 5. Produces plots of
%   (i) position vs. time
% for the response of vehicle, with mass m, stiffness k,
% and damping parameter b, when the vehicle starts from initial conditions
% y = 0 and v = 0 under the action of a suddenly applied force
% fa = 1000 pounds ( 4448 Newtons) at t = 0.

clear variables
global m k b fa
% Input parameters
m = 1134;
k = input('Enter the stiffness "k" in Newtons/meter ' );
b = 19930;
fa= 4448;
% Input initial conditions.
y0= 0;
v0= 0;
tspan = input('Enter the duration "T" of the desired time history, in seconds ');
X0 = [ y0 ; v0 ];
% Integrate equations of motion
[t,X] = ode45('EqHW6Pr2', tspan, X0);
% Plot results
plot(t,X(:,1)), title('Time History of Displacement'),
xlabel('Time  [Seconds]'), ylabel('Displacement  [meters]'), pause
```

The adaptation of 'pl_on_spr.m' involved only the name change to 'EqHW6Pr2.m'.

```
% 'EqHW6Pr2.m'  Adapted from 'pl-on-spr.m'. Provides equation of motion
% for vehicle with unblocked or blocked springs
% to be integrated by script 'HW6Pr2.m'
function Xdot = EqHW6Pr2(t,X)

global m k b fa

Xdot = [ 0  1 ; -k/m -b/m ]*X + [ 0 ; fa/m ];
```

Using these scripts with the stiffness inputs $k_u = 87560$ N/m and $k_b = 175100$ N/m, and an integration time of 1.0 second produced the response curve shown in Fig.2 for the unblocked vehicle, and the response curve shown in Fig.3 for the blocked vehicle. The two responses are compared in Fig.4. Note the quicker response to a smaller deflection for the blocked vehicle.

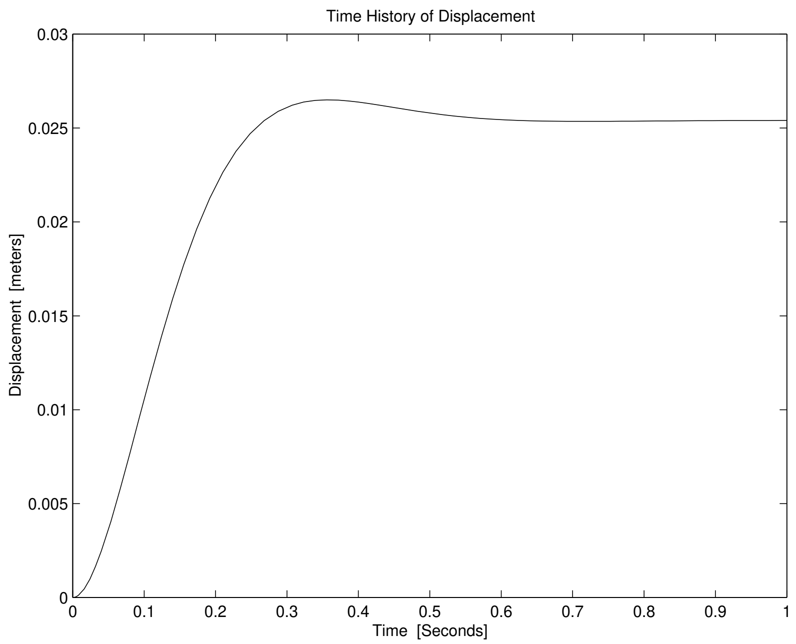

Figure 3: Response of Unblocked Vehicle

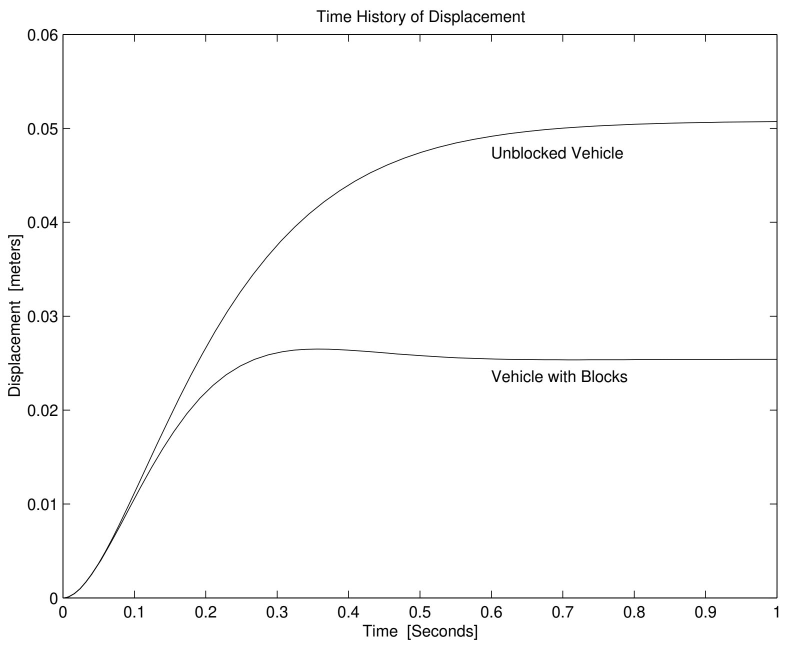

Figure 4: Comparison of Responses

3. (44 points, Part a, b: 6 points each, Part c - f: 3 points each, Part g - j: 5 points each) The sketch in Fig.5 shows the linear model used to study crashworthiness. The mass m of the vehicle is (2000 pounds)(0.4536 kg/pound) = 907.2 kg and the vehicle speed at impact v_o is (10 mi/hr)(5280 ft/mi)(0.3048 meters/ft)/(3600 secs/hr) =

4.470 meters/sec. When there is no energy dissipation ($b = 0$) the displacement of the mass goes through a half-cycle of undamped vibration of the form

$$x(t) = A \sin \omega_o t$$

The vehicle then retreats from the barrier with velocity $-v_o$. The amplitude A of the oscillation is given as 6 inches which is (6 in)(0.0254 meters/in) = 0.1524 meters, but the undamped natural frquency ω_o is unknown.

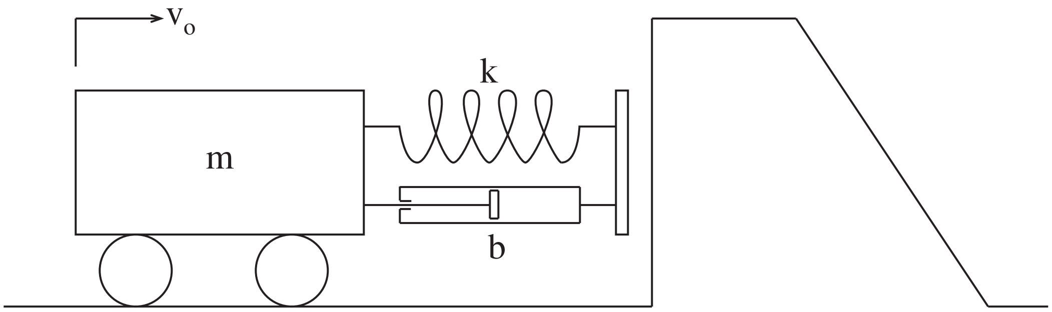

Figure 5: Model of Vehicle Impacting a Barrier

One way to calculate ω_o is to set the velocity obtained by differentiating $x(t)$ equal to v_o at $t = 0$. This leads to

$$\omega_o = \frac{v_o}{A} = \frac{4.470}{0.1524} = 29.33 \text{ rad/sec}$$

Another way to obtain ω_o is to use conservation of energy,

$$KE = \frac{1}{2} m v_o^2 = \frac{1}{2} k A^2$$

to get k, and then use $\omega_o^2 = k/m$ to evaluate ω_o.

(a) If the value of ω_o is already known, then the effective stiffness k is obtained from

$$k = m\omega_o^2 = (907.2)(29.33)^2 = 780,400 \text{ N/m}$$

If ω_o is not available, then the stiffness can be obtained directly from conservation of energy.

$$k = m \left(\frac{v_o}{A}\right)^2 = 907.2 \left(\frac{4.470}{0.1524}\right)^2 = 780,500 \text{ N/m}$$

(b) During the half-cycle of undamped sinusoidal motion $x(t) = A \sin \omega_o t$ the acceleration is $d^2x/dt^2 = -A\omega_o^2 \sin \omega_o t$. The peak deceleration occurs at the instant of peak deflection of the spring and has the magnitude

$$Decel_{peak} = A\omega_o^2 = (0,1524)(29.33)^2 = 131.1 \text{ meters/sec}^2$$

The geometric compatibility requirements are satisfied by taking x to represent the displacement of the mass *and* the deformation of the spring, and $v = dx/dt$ to represent the velocity of the mass *and* the relative velocity across the damper. The constitutive equations are

$$f_m = m\frac{dv}{dt} \qquad f_d = bv \qquad \text{and} \qquad f_k = kx$$

and the force-balance requirement is

$$f_m = -f_d - f_k$$

These model requirements can be used to construct the state equations for the state variables x and v

$$\frac{d}{dt}\left\{\begin{array}{c} x \\ v \end{array}\right\} = \left[\begin{array}{cc} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{array}\right]\left\{\begin{array}{c} x \\ v \end{array}\right\}$$

The desired outputs are (i) the deflection $disp = x$ of the fender, (ii) the deceleration of the vehicle $decel = (k/m)x + (b/m)v)$, and (iii) the total force on the barrier $f_{barr} = kx + bv$. In matrix notation

$$\left\{\begin{array}{c} disp \\ decel \\ f_{barr} \end{array}\right\} = \left[\begin{array}{cc} 1 & 0 \\ k/m & b/m \\ k & b \end{array}\right]\left\{\begin{array}{c} x \\ v \end{array}\right\}$$

The state equations can be integrated by adapting the scripts 'POS.m' and 'pL_on_spr.m' for the Plate on Springs problem of Assignment 5 to apply to the present problem. The adaptation of 'POS.m' involved inputting the specific values of m, k, x_o, and v_o and letting the program ask for the desired value of ζ instead of the desired value of b. The adapted version of 'POS.m' is called 'HW6Pr3.m'. The script for 'HW6Pr3.m' follows:

```
% HW6Pr3.m MATLAB Script adapted from POS.m which was
% MATLAB script for Problem 3 in 2.003 Assignment 5. Produces plots of
%   (i) displacement vs. time
% (ii) deceleration vs. time
% (iii) force on barrier vs. time
% for the response of vehicle, with mass m= 907.2 kg,
% stiffness k = 780500 N/m, and damping ratio zeta,
% when the vehicle impacts barrier with initial conditions
% x = 0 and v = 4.470 m/s  at t = 0.

clear variables
```

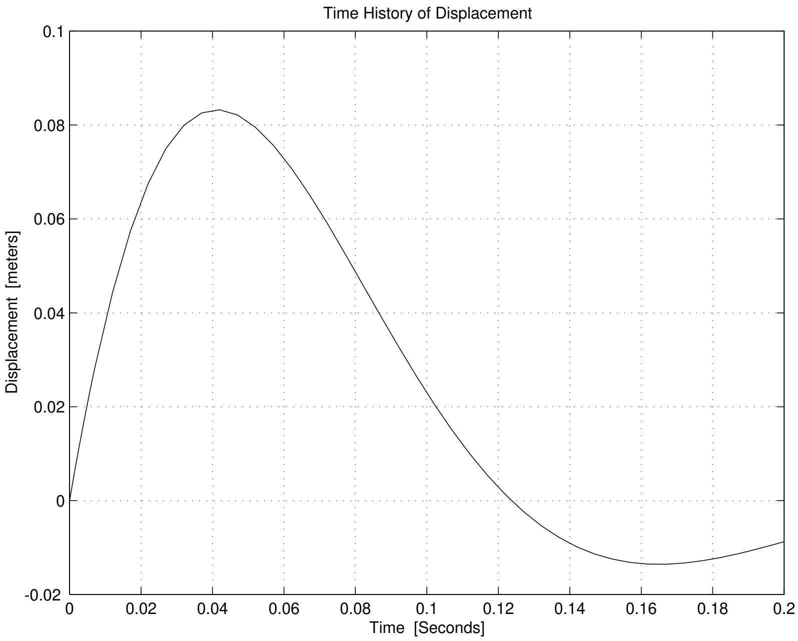

Figure 6: Displacement Response for $\zeta = 0.05$

```
global m k b
% Input parameters
m = 907.2;
k = 780500;
zeta = input('Enter the damping ratio, zeta ');
b = 2*zeta*29.33*907.2;
% Input initial conditions.
x0= 0;
v0= 4.470;
tspan = input('Enter the duration "T" of the desired time history, in seconds ');
X0 = [ x0 ; v0 ];
% Integrate equations of motion
[t,X] = ode45('EqHW6Pr3', tspan, X0);
% Obtain outputs
disp = X(:,1);
decel = (k/m)*X(:,1) + (b/m)*X(:,2);
fbarr = m*decel;
% Plot results
plot(t,disp), title('Time History of Displacement'), grid,
xlabel('Time  [Seconds]'), ylabel('Displacement  [meters]'), pause
plot(t,decel), title('Time History of Deceleration'), grid,
xlabel('Time  [Seconds]'), ylabel('Deceleration  [meters/sec sqd]'),pause
plot(t, fbarr), title('Time History of Force on Barrier'), grid
xlabel('Time  [Seconds]'), ylabel('Force  [Newtons]'),
```

The script for the equation called for by 'HW6Pr3' is called 'EqHW6Pr3.m'. It is similar to 'EqHW6Pr2' except that here there is no applied force f_a.

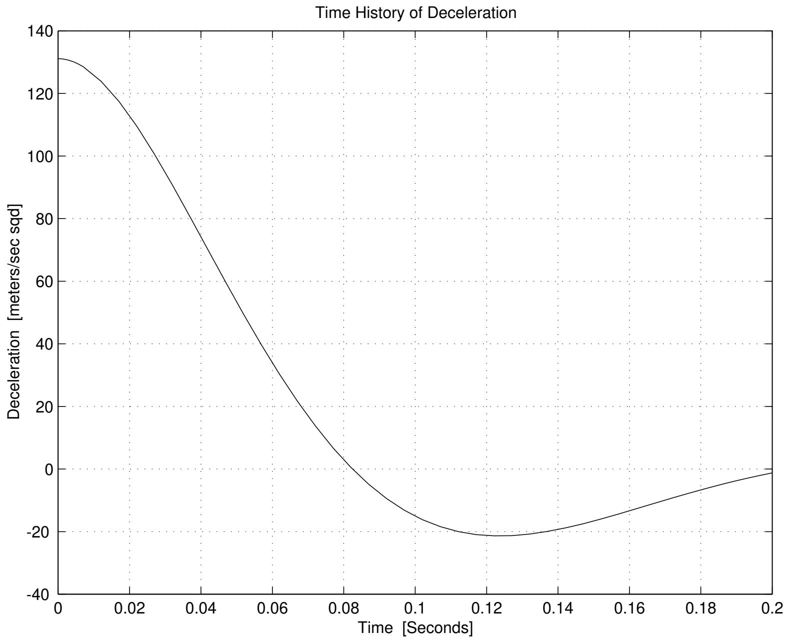

Figure 7: Deceleration Response for $\zeta = 0.05$

```
% 'EqHW6Pr3.m'  Adapted from 'pl-on-spr.m'. Provides equation of motion
% for vehicle impacting barrier
% to be integrated by script 'HW6Pr3.m'
function Xdot = EqHW6Pr3(t,X)

global m k b

Xdot = [ 0  1 ; -k/m -b/m ]*X ;
```

Typical plots produced by 'HW6Pr3.m' are shown in Figs. 6-8, which were obtained for $\zeta = 0.5$.

(g) The plots obtained are only valid during the interval in which the fender remains in contact with the barrier. In the undamped case, this period was a half cycle of sinusoidal oscillation. In the damped cases, the model results are valid only as long as the barrier force is compressive. If tension is required the fender

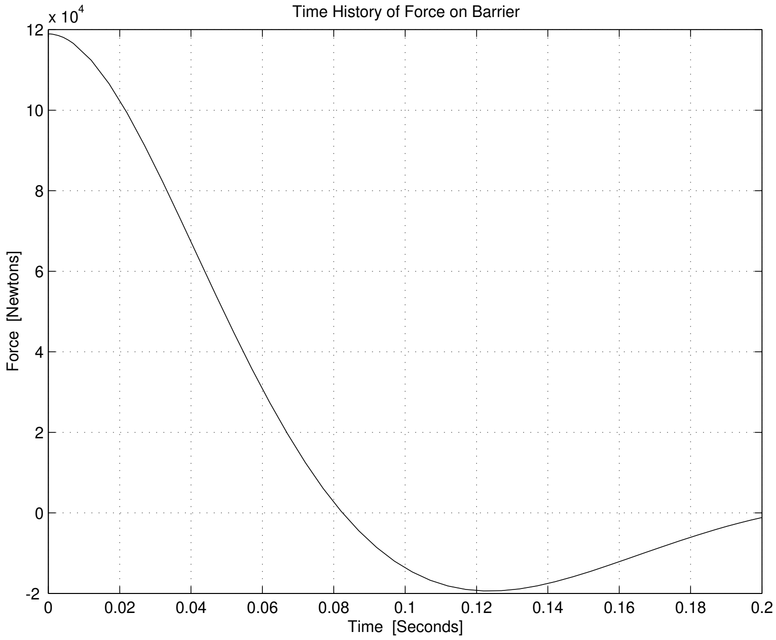

Figure 8: Barrier Force Response for $\zeta = 0.05$

separates from the barrier. For the case $\zeta = 0.5$ we see that the model predicts tensile barrier force after $t = 0.082$ seconds. In *all four* cases there is separation of the vehicle from the barrier after the impact.

(h) Peak decelerations are proportional to peak barrier forces (this is a consequence of Newton's law $F = ma$). In the undamped case, the barrier force is delivered by the spring. Its magnitude is $f_{barr,u} = mA\omega_o^2 = mv_o\omega_o$. In the damped case, the barrier force is delivered by both the spring and the damper. At the initial instant of impact, when the fender begins to move with velocity v_o, but hasn't yet deflected the spring, the entire barrier force is delivered by the damper. Its magnitude is $f_{barr,d} = bv_o$ which can be made as large as desired by increasing the parameter b. This initial damping force will be the *same* as the peak undamped force if $bv_o = mv_o\omega_o$. This occurs when $\zeta = 0.5$.

(i) The peak deceleration is greatest in Case (f), $\zeta = 1.0$.

(j) By trial and error with 'HW6Pr3' one finds that 'decel' has a fairly flat minimum value of slightly more than 106 meters/second^2 over the range $0.25 < \zeta < 0.30$.