2.003 Fall 1999 Solution of Homework Assignment 7

1. **The start-up transient of a CD player.** (20 points, 5 points each)

   (a) Let the outer radius of the CD be R, and let the radius of the hole be r. Let the mass density of the disc be ρ per unit area, and let the total mass of the CD be m. Then

   $$m = \rho(\pi R^2) - \rho(\pi r^2) = \pi\rho(R^2 - r^2)$$

   The moment of inertia is

   $$I_{CD} = \rho(\pi R^2)(\frac{1}{2}R^2) - \rho(\pi r^2)(\frac{1}{2}r^2) = \frac{1}{2}\pi\rho(R^4 - r^4)$$

   and

   $$\frac{1}{2}\pi\rho(R^4 - r^4) = \frac{1}{2}\pi\rho(R^4 - r^4)\left[\frac{m}{\pi\rho(R^2 - r^2)}\right] = \frac{1}{2}m(R^2 + r^2)$$

   Substitute $R = 0.060$ m, $r = 0.0075$ m, and $m = 0.028$ kg to get $I_{CD} =$ 5.12e-5 kg-m^2.

   (b) At steady state, the motor torque T_m is balanced by the frictional torque $T_{fric} = B_m\omega_{ss}$. If ω_{ss} is the same for the disc mounted and not mounted, the damping coefficient B_m must also be the same in both cases. In the transient, it is necessary to consider the complete dynamic analysis. The constitutive equations are

   $$T_I = I\frac{d\omega_m}{dt} \qquad \text{and} \qquad T_{fric} = B_m\omega_m$$

   where I is the total moment of inertia of the rotating parts, and the torque balance is

   $$T_I = T_m - T_{fric}$$

   which lead to the differential equation

   $$I\frac{d\omega_m}{dt} + B_m\omega_m = T_m \qquad \text{or} \qquad \left(\frac{I}{B_m}\right)\frac{d\omega_m}{dt} + \omega_m = \frac{T_m}{B_m}$$

   from which we learn that the decay time-constant is $\tau = I/B_m$. Now we are given the time it takes the spindle to reach 95% of ω_{ss} which is known to be 3 time-constants, for the spindle alone, and for spindle plus the mounted CD, so

   $$\tau_{noCD} = \frac{I_r}{B_m} = \frac{0.3}{3} \qquad \text{and} \qquad \tau_{withCD} = \frac{I_r + I_{CD}}{B_m} = \frac{2.0}{3}$$

   and, by subtraction to eliminate I_r, we find

   $$\frac{I_{CD}}{B_m} = \frac{2.0 - 0.3}{3} \qquad \text{or} \qquad B_m = \frac{3I_{CD}}{1.7} = \text{9.04e-5 N-m-s/r}$$

   This is the answer for both parts (i) and (ii).

(c) To estimate I_r, return to τ_{noCD} and solve for I_r

$$I_r = B_m \tau_{noCD} = 9.04\text{e-}5 \frac{0.3}{3} = 9.04\text{e-}6 \text{ kg-m}^2$$

(d) The standard form for state-determined representation of a dynamic system is

$$\begin{aligned} \tfrac{d}{dt}\mathbf{x} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned}$$

where $\mathbf{x}$ is a column matrix of the *state* variables, $\mathbf{u}$ is a column matrix of the *input* variables, and $\mathbf{y}$ is a column matrix of the desired *output* variables. The parameters which govern the natural response of the system are contained in the square matrix **A**. The parameters which describe how the input is delivered are contained in **B**, the parameters which describe how the output depends on the state variables are contained in **C**, and the parameters which describe how the output depends directly on the input are contained in **D**.

In the present case there is only one state variable ω_m, one input T_m, and one output ω_m, so all the matrices reduce to scalars. The state equation for ω_m is

$$\frac{d\omega_m}{dt} = -\frac{B_m}{I_{total}}\omega_m + \frac{1}{I_{total}}T_m$$

where $I_{total} = I_r + I_{CD} = (0.904 + 5.12)\text{e-}5 = 6.02\text{e-}5$ kg-m^2. The matrices of the standard form reduce to the following scalars:

$$\mathbf{x} = \omega_m, \quad \mathbf{A} = -\frac{B_m}{I_{total}}, \quad \mathbf{u} = T_m, \quad \mathbf{B} = \frac{1}{I_{total}}, \quad \mathbf{y} = \omega_m, \quad \mathbf{C} = 1, \quad \mathbf{D} = 0$$

2. **Speed control of CD player.** (25 points, 5 points each)

(a) The dimensions of the parameter G are [torque]/[angular velocity], or [FLT], or [ML^2/T]. In the SI system, the common unit is the Newton-meter per radian per second, or N-m-s.

(b) Assume that $\omega_{sensed} = \omega_m$, and insert $T_m = G(r - \omega_m)$ in the state equation in Problem 1(d) to get

$$\frac{d\omega_m}{dt} = -\frac{B_m + G}{I_{total}}\omega_m + \frac{G}{I_{total}}r$$

as the state equation for the controlled system. Comparing this equation for the controlled system with the equation in 1(d) above, note that the input **u** has changed from a torque T_m to a speed r, and that, **A** has changed from $-B_m/I_{total}$ to $-(B_m + G)/I_{total}$, and **B** has changed from $1/I_{total}$ to G/I_{total}. The system is still a first-order dynamic system but its behavior can easily be modified by adjusting the gain G.

(c) The steady-state speed corresponding to 4 revolutions/second is $\omega_{ss} = 8\pi = 25.1$ radians/second.

(i) For the uncontrolled system, the motor torque required to maintain this speed is

$$T_m = B_m \omega_{ss} = (9.04\text{e-}5)(8\pi) = 2.27\text{e-}3 \text{ N-m}$$

(ii) For the controlled system, the reference speed required to maintain the same speed is

$$r = \frac{B_m + G}{G}\omega_{ss} = \frac{9.04\text{e-}5 + G}{G}25.1$$

(d) The decay time-constant for the uncontrolled system is $\tau_u = I_{total}/B_m$, while the decay time-constant for the controlled systgem is $\tau_c = I_{total}/(B_m + G)$. If the controlled system is to be five times faster than the uncontrolled system, it is necessary for $\tau_c = \tau_u/5$, or

$$\frac{I_{total}}{B_m + G} = \frac{1}{5}\frac{I_{total}}{B_m}$$

from which we obtain

$$G = 4B_m = 4(9.04\text{e-}5) = 36.2\text{e-}5 \text{ N-m-s}$$

(e) The scripts below were adapted from the scripts 'car.m' and 'car_visc.m' of Assignment 1 to fit the state equation in Problem 2(b). The symbol v in the script stands for the angular velocity ω_m. After the equation is integrated to obtain $\omega_m(t)$, the motor torque T_m is constructed from the control algorithm

$$T_m = G(r - \omega_m)$$

and plotted. See Fig.1.

Note that the maximum motor torque occurs at $t = 0$ when $\omega_m = 0$. The magnitude of the maximum torque is

$$T_m(0) = Gr = (B_m + G)\omega_{ss} = (B_m + 4B_m)\omega_{ss} = 5B_m\omega_{ss} = 11.35\text{e-}3 \text{ N-m}$$

Note that the five-fold increase in response speed is accompanied by a five-fold increase in maximum torque.

Script for MATLAB m-file 'VelFdbk.m'

```
% VelFdbk.m
% This program calculates the torque in CD player with velocity feedback.
% The differential equation used to model the system is:
%
% Itot dv/dt =  - (Bm + G) v + G r
%
```

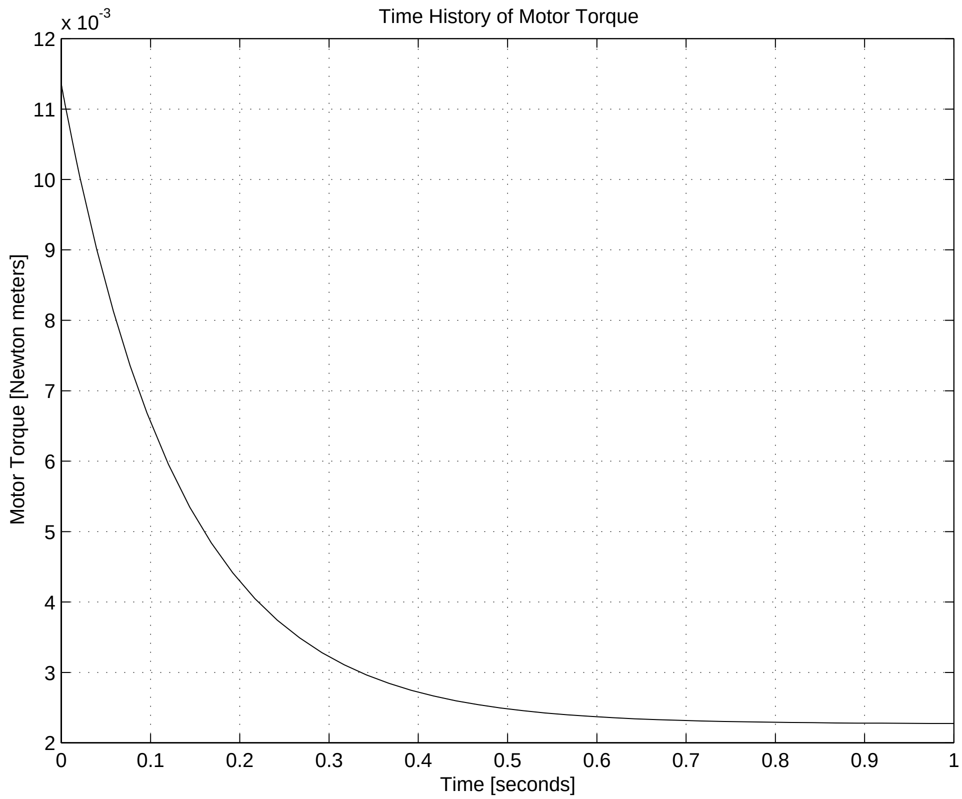

Figure 1: Transient Motor Torque under Velocity Feedback Control

```
% This equation is stored in a separate m-file 'EqVelFdbk.m' called by this
% program (a tedious MATLAB detail).
%
clear variables
%
% Declare global parameters (a MATLAB detail).
%
global Itot Bm r G
%
% Input parameter values.
%
Itot = input('Enter the total moment of inertia "I(CD) + I(rotor)" in SI units: ');
Bm =  input('Enter the linear friction coefficient Bm in SI units: ');
ss = input('Enter the steady-state speed in rad/sec ');
G =  input('Enter the value of the gain G in SI units: ');
%
% Input initial condition.
%
v0 = input('Enter the initial angular velocity in rad/sec: ');
%
```

```
% Input integration time.
%
tspan = input('Enter time interval of integration in seconds: ');
%
% Calculate r
%
r= (( Bm + G ) / G) * ss;
%
% Call a numerical integration algorithm.
%
[t,v] = ode45('EqVelFdbk', tspan, v0);
%
%   Calculate motor torque
%
Tm = G * ( r - v );
% Plot the results.
%
plot(t,Tm)
xlabel('Time [seconds]')
ylabel('Motor Torque [Newton meters]')
title('Time History of Motor Torque')
grid\\
```

Script for MATLAB m-file 'EqVelFdbk.m'

```
%  EqVelFdbk.m
% This function, which is called by the program 'VelFdbk.m', contains the
% equation of motion of a CD player with velocity feedback.
%
%  Introduce  V_dot, the time derivative of v.
%
function V_dot = EqVelFdbk(t,v)
%
% Declare global parameters (a MATLAB detail).
%
global Itot Bm r G
%
% The mathematical model of velocity feedback applied to the CD player
% provides an equation for V_dot.
%
V_dot = -((Bm + G)/Itot)*v + (G / Itot) * r;
%
% That's all there is to it!
```

3. **Locked Load Response.** (25 points, 5 points each) With the load clamped so that it cannot rotate, the application of motor torque causes twisting of the coupler and oscillation of the rotor. The system consists of the motor rotor with moment of inertia I_r and the coupler with linear torsional stiffness K_c and linear damping with damping coefficient B_c, acted on by the motor torque T_m and a frictional motor torque with linear damping coefficient B_m, as shown in Fig. 2.

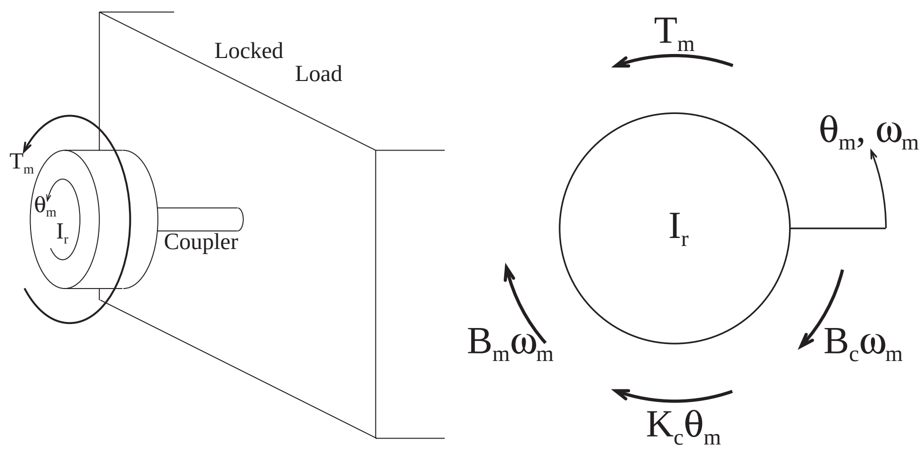

Figure 2: Motor Rotor connected to Locked Load by Elastic Coupler

(a) The fundamental requirements are:
Geometric compatibility: $\omega_m = d\theta_m/dt$
Constitutive equations: $T_I = I_r d\omega_m/dt, \quad T_f = (B_m + B_c)\omega_m, \quad T_k = K_m\theta_m$
Torque balance: $T_I = T_m - T_f - T_k$
Take θ_m and ω_m as state variables. Then one equation is provided by the geometric compatibility requirement. The second equation is obtained by substituting the constitutive equations into the torque-balance requirement to get

$$I_r \frac{d\omega_m}{dt} = T_m - (B_m + B_c)\omega_m - K_c\theta_m$$

With T_m as the input and θ_m as the desired output, the standard form for a state-determined system

$$\begin{aligned} \tfrac{d}{dt}\mathbf{x} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} + \mathbf{Du} \end{aligned}$$

becomes

$$\frac{d}{dt}\left\{ \begin{array}{c} \theta_m \\ \omega_m \end{array} \right\} = \left[\begin{array}{cc} 0 & 1 \\ -\frac{K_c}{I_r} & -\frac{B_m+B_c}{I_r} \end{array} \right] \left\{ \begin{array}{c} \theta_m \\ \omega_m \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ \frac{1}{I_r} \end{array} \right\} T_m$$

with output θ_m given by

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} = \{\ 1 \quad 0\ \} \left\{ \begin{array}{c} \theta_m \\ \omega_m \end{array} \right\} + 0 = \theta_m$$

(b) The behavioral parameters ω_o and ζ appear in the matrix $\mathbf{A}$ of second-order systems in the following pattern

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & -2\zeta\omega_o \end{bmatrix} \qquad (1)$$

By comparing the matrix $\mathbf{A}$ in the equation for the locked-load response with (1) we identify the the following relationships for the behavioral parameters

$$\omega_o^2 = \frac{K_c}{I_r} \qquad \text{and} \qquad 2\zeta\omega_o = \frac{B_m + B_c}{I_r}$$

which lead to

$$\omega_o = \sqrt{\frac{K_c}{I_r}} = \sqrt{\frac{1.24\text{e-}2}{5\text{e-}5}} = 15.75 \text{ rad/sec} \quad \text{and} \quad \zeta = \frac{B_m + B_c}{2\sqrt{K_c I_r}} = \frac{1\text{e-}4 + 2\text{e-}5}{2\sqrt{(1.24\text{e-}2)(5\text{e-}5)}} = 0.0762$$

(i) The damped natural frequency is

$$\omega_d = \omega_o\sqrt{1-\zeta^2} = 15.75\sqrt{1-(0.0762)^2} = 15.70 \text{ rad/sec}$$

(ii) The decrement ratio for a damped sinusoidal oscillation, as defined in the Notes for Lecture 6, is the ratio of the amplitudes of two successive peaks. The time increment between two such peaks is half of the period of the damped oscillation. The envelope of the damped oscillation decays in proportion to $\exp(-\zeta\omega_o t)$, so with $t = \frac{1}{2}T_d = \pi/\omega_d$ the decrement ratio is

$$\text{dec ratio} = \exp(-\zeta\omega_o\pi/\omega_d) = \exp(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}) = \exp(-\frac{0.0762\pi}{\sqrt{1-(0.0762)^2}}) = 0.787$$

(c) The dimensions of G_1 are [torque/angular velocity]. Its units are N-m/r/s or N-m-s/r in the SI system of units. When the control algorithm (with $\omega_{sensed} = \omega_m$),

$$T_M = G_1(r_1 - \omega_m),$$

is substituted in the state equations of the uncontrolled locked-load system, the resulting equations for the velocity-controlled system are

$$\frac{d}{dt}\left\{ \begin{array}{c} \theta_m \\ \omega_m \end{array} \right\} = \begin{bmatrix} 0 & 1 \\ -\frac{K_c}{I_r} & -\frac{B_m + B_c + G_1}{I_r} \end{bmatrix} \left\{ \begin{array}{c} \theta_m \\ \omega_m \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ \frac{G_1}{I_r} \end{array} \right\} r_1$$

The input is changed from the motor torque T_m to the reference speed r_1, and the Matrices $\mathbf{A}$ and $\mathbf{B}$ now depend on the gain G_1. The undamped natural frequency ω_o remains unchanged but the damping ratio now depends on the gain G_1. By comparing the matrix $\mathbf{A}$ with (1) we identify

$$2\zeta\omega_o = \frac{B_m + B_c + G_1}{I_r}$$

To obtain the value of G_1 required to yield a critical;ly damped response, set $\zeta = 1$ and solve for G_1

$$G_1 = 2I_r\omega_o - B_m - B_c = 2(5\text{e-}5)(15.75) - 1\text{e-}4 - 2\text{e-}5 = 14.55\text{e-}4 \text{ N-m-s/r}$$

(d) The dimensions of G_2 are [torque/angle]. Its units are N-m in the SI system of units. When the control algorithm (with $\theta_{sensed} = \theta_m$),

$$T_M = G_2(r_2 - \theta_m),$$

is substituted in the state equations of the uncontrolled locked-load system, the resulting equations for the position-controlled system are

$$\frac{d}{dt}\left\{\begin{array}{c}\theta_m\\ \\ \omega_m\end{array}\right\} = \left[\begin{array}{cc}0 & 1\\ \\ -\frac{K_c+G_2}{I_r} & -\frac{B_m+B_c}{I_r}\end{array}\right]\left\{\begin{array}{c}\theta_m\\ \\ \omega_m\end{array}\right\} + \left\{\begin{array}{c}0\\ \\ \frac{G_2}{I_r}\end{array}\right\} r_2$$

By comparing the matrix **A** in this equation with (1), we identify the following relationships for the behavioral parameters

$$\omega_o^2 = \frac{K_c + G_2}{I_r} \qquad \text{and} \qquad 2\zeta\omega_o = \frac{B_m + B_c}{I_r}$$

Both the undamped natural frequency ω_o and the damping ratio ζ now depend on the gain G_2.

(i) The undamped natural frequency is

$$\omega_o = \sqrt{\frac{K_c + G_2}{I_r}}$$

(ii) The damping ratio is

$$\zeta = \frac{B_m + B_c}{2I_r\omega_o} = \frac{B_m + B_c}{2\sqrt{(B_m + G_2)I_r}}$$

(e) The control algorithms can be implemented by ordinary mechanical elements connected between a desired-motion driver and the motor rotor.

(i) The velocity feedback control algorithm

$$T_m = G_1(r_1 - \omega_m)$$

states that a torque is applied to the rotor which is proportional to the difference in speeds between the motion driver and the rotor. A linear *friction element* connected between the motion driver and the rotor would apply this same torque if its damping coefficient were G_1.

(ii) The position feedback control algorithm

$$T_m = G_2(r_2 - \theta_m)$$

states that a torque is applied to the rotor which is proportional to the difference in angular position between the motion driver and the rotor. A linear *torsional spring* connected between the motion driver and the rotor would apply this same torque if its stiffness were G_2.

A drawback of these mechanical implementations is the necessity of constructing a *mechanical* motion driver to provide the information about the desired motion.

4. **Third order dynamic system.** (30 points, 10 points for part a, 5 points each for part b through e) Symmetrical mechanical drive. See Fig.3. Two equal inertias I with equal linear friction coefficients B are coupled by a shaft with torsional stiffness K_c and torsional damping coefficient B_c.

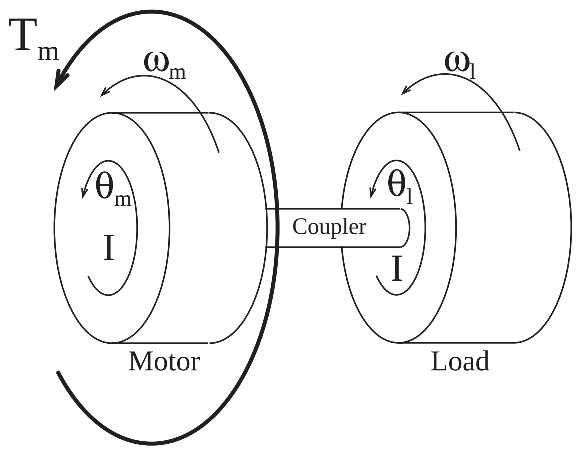

Figure 3: Symmetrical Mechanical Drive

(a) The fundamental requirements are:
Geometrical Compatibility: $\Delta\theta = \theta_m - \theta_l, \quad d\Delta\theta/dt = \omega_m - \omega_l$

Constitutive Equations: $T_k = K_c\Delta\theta, \quad T_d = B_c d\Delta\theta/dt, \quad T_{f,m} = B\omega_m,$
$T_{f,l} = B\omega_l, \quad T_{I,m} = I d\omega_m/dt, \quad T_{I,l} = I d\omega_l/dt.$

Torque Balance: Motor: $T_{I,m} = T_m - K_c\Delta\theta - B\omega_m - B_c d\Delta\theta/dt,$
Load: $T_{I,l} = K_c\Delta\theta - B\omega_l + B_c d\Delta\theta/dt$

In the following state equations , the first equation expresses the geometric compatibility requirement, the second equation is the result of inserting the constitutive equations into the torque balance for the motor, and the third is the result of substituting the constitutive equations into the torque balance for the load.

$$\frac{d}{dt}\left\{\begin{array}{c}\Delta\theta\\ \omega_m\\ \omega_l\end{array}\right\} = \left[\begin{array}{ccc}0 & 1 & -1\\ -\frac{K_c}{I} & -\frac{B+B_c}{I} & \frac{B_c}{I}\\ \frac{K_c}{I} & \frac{B_c}{I} & -\frac{B+B_c}{i}\end{array}\right]\left\{\begin{array}{c}\Delta\theta\\ \omega_m\\ \omega_l\end{array}\right\} + \left\{\begin{array}{c}0\\ \frac{1}{I}\\ 0\end{array}\right\}T_m$$

(b) The new variables $\Delta\omega$ and ω_{avg} satisfy the following equations:

$$\begin{aligned} 2\omega_{avg} &= \omega_m + \omega_l \\ \Delta\omega &= \omega_m - \omega_l \end{aligned}$$

By adding and subtracting these equations we find

$$\begin{aligned} \omega_m &= \omega_{avg} + \Delta\omega/2 \\ \omega_l &= \omega_{avg} - \Delta\omega/2 \end{aligned}$$

(c) We can construct the state equations for the new variables $\Delta\theta$, $\Delta\omega$ and ω_{avg} one at a time. For $\Delta\theta$, we have

$$\frac{d\Delta\theta}{dt} = \omega_m - \omega_l = \Delta\omega$$

For $\Delta\omega$, we write

$$\begin{aligned} \frac{d\Delta\omega}{dt} = \frac{d\omega_m}{dt} - \frac{d\omega_l}{dt} &= \left[-\frac{K_c}{I}\Delta\theta - \frac{B+B_c}{I}(\omega_{avg}+\Delta\omega/2) + \frac{B_c}{I}(\omega_{avg}-\Delta\omega/2)\right] + \frac{T_m}{I} \\ &\quad - \left[\frac{K_c}{I}\Delta\theta - \frac{B+B_c}{I}(\omega_{avg}-\Delta\omega/2) + \frac{B_c}{I}(\omega_a vg+\Delta\omega/2)\right] \\ &= -\frac{2K_c}{I}\Delta\theta - \frac{B+2B_c}{I}\Delta\omega + \frac{T_m}{I} \end{aligned}$$

Similarly, for ω_{avg}, we write

$$\begin{aligned} \frac{d\omega_{avg}}{dt} = \frac{1}{2}(\frac{d\omega_m}{dt} + \frac{d\omega_l}{dt}) &= \frac{1}{2}\left[-\frac{K_c}{I}\Delta\theta - \frac{B+B_c}{I}(\omega_{avg}+\Delta\omega/2) + \frac{B_c}{I}(\omega_{avg}-\Delta\omega/2)\right] + \frac{T_m}{I} \\ &\quad + \frac{1}{2}\left[\frac{K_c}{I}\Delta\theta - \frac{B+B_c}{I}(\omega_{avg}-\Delta\omega/2) + \frac{B_c}{I}(\omega_a vg+\Delta\omega/2)\right] \\ &= -\frac{B}{I}\omega_{avg} + \frac{T_m}{I} \end{aligned}$$

These state equations for the new variables can be arranged in matrices of a third-order sytem:

$$\frac{d}{dt}\left\{\begin{array}{c}\Delta\theta\\ \Delta\omega\\ \omega_{avg}\end{array}\right\} = \left[\begin{array}{ccc}0 & 1 & 0\\ -2\frac{K_c}{I} & -\frac{B+2B_c}{I} & 0\\ 0 & 0 & -\frac{B}{I}\end{array}\right]\left\{\begin{array}{c}\Delta\theta\\ \Delta\omega\\ \omega_{avg}\end{array}\right\} + \left\{\begin{array}{c}0\\ \frac{1}{I}\\ \frac{1}{I}\end{array}\right\}T_m$$

or, as a combination of a second-order sub-system

$$\frac{d}{dt}\left\{\begin{array}{c}\Delta\theta\\ \\ \Delta\omega\end{array}\right\}=\left[\begin{array}{cc}0 & 1\\ \\ -2\frac{K_c}{I} & -\frac{B+2B_c}{I}\end{array}\right]\left\{\begin{array}{c}\Delta\theta\\ \\ \Delta\omega\end{array}\right\}+\left\{\begin{array}{c}0\\ \\ \frac{1}{I}\end{array}\right\}T_m$$

describing the relative motion of the motor and the load, and a first-order sub-system

$$\frac{d\omega_{avg}}{dt}=-\frac{B}{I}\omega_{avg}+\frac{1}{I}T_m$$

describing the average speed of motor and load. Both sub-systems have the motor torque as input, but the second-order sub-system is independent of the average speed ω_{avg}, and the first-order sub-system is independent of the *relative* motion between the motor and the load.

(d) The first-order sub-system equation can be rewritten as

$$\frac{I}{B}\frac{d\omega_{avg}}{dt}+\omega_{avg}=\frac{T_m}{B}$$

from which we identify the *decay time-constant*

$$\tau=\frac{I}{B}=\frac{\text{5e-5}}{\text{1.0e-4}}=0.5\text{ sec}$$

and the *steady-state speed*

$$\omega_{ss}=\frac{T_m}{B}=\frac{\text{6e-3}}{\text{1.0e-4}}=60\text{ rad/sec}$$

(e) From the second-order sub-system equations we identify

$$\omega_o^2=\frac{2K_c}{I}\qquad\text{and}\qquad 2\zeta\omega_o=\frac{B+2B_c}{I}$$

from which we obtain the behavioral parameters of the second-order sub-system

$$\omega_o=\sqrt{\frac{2(\text{1.24e-2})}{\text{5e-5}}}=22.3\text{ rad/sec}\qquad\text{and}\qquad\zeta=\frac{\text{1.0e-4}+2(\text{2e-5})}{2(22.3)(\text{5e-5})}=0.0629$$