2.003 Fall 1999 Solution of Homework Assignment 8

1. **Sliding Mass.** (25 points. 4 points each for parts a, b, d, and f. 6 points for part c, 3 points for part e) A mass m is subjected to an accelerating force f and a retarding friction force, modeled linearly as bv.

(a) The equation of motion is

$$m\frac{dv}{dt} + bv = f \qquad \text{or} \qquad \frac{dv}{dt} = -\frac{b}{m}v + \frac{1}{m}f \tag{1}$$

(b) The steady-state velocity, when $f = 10$ Newtons, and $m = 1000$ kg, and $b = 100$ N/m/s is

$$v_{ss} = \frac{f}{b} = \frac{10}{100} = 0.10 \text{ m/s}$$

(c) Consider the input $f = f_a \sin \Omega t$ to be the imaginary part of the complex input $f_a \exp(i\Omega t)$ and look for a complex solution of the form $v = A\exp(i\Omega t)$. After substitution in (1) we find that this is indeed a solution , if

$$A = \frac{f_a}{b + im\Omega} = \frac{f_a}{b}\frac{1}{\sqrt{1 + (\frac{m\Omega}{b})^2}} \exp(i\phi)$$

where the phase angle ϕ is fixed by the relation

$$\tan\phi = -\frac{m\Omega}{b}$$

The steady-state solution to the input $f = f_a \sin \Omega t$ is then the imaginary part of the complex solution $A\exp(i\Omega t)$, which is

$$v = \frac{f_a}{b}\frac{1}{\sqrt{1 + (\frac{m\Omega}{b})^2}} \sin(\Omega t + \phi)$$

The magnitude of the response amplitude is

$$M(\Omega) = \frac{f_a}{b}\frac{1}{\sqrt{1 + (\frac{m\Omega}{b})^2}}$$

and the phase angle is

$$\phi = -\arctan\frac{m\Omega}{b}$$

For the case where $f_a = 10$ Newtons, the values of $M(\Omega)$ and $\phi(\Omega)$ for the frequencies $\Omega = 0.5$ rad/sec, $\Omega = 0.05$ rad/sec, and $\Omega = 0.005$ rad/sec are

(i) $\mathrm{M}(0.5) = \frac{10}{100}\frac{1}{\sqrt{1+(\frac{(1000)(0.5)}{100})^2}} = 0.01961$ m/s,

$\phi(0.5) = \arctan[(1000)(0.5)/100] = 1.373r = 78.7\,\mathrm{deg}$

(ii) $\mathrm{M}(0.05) = \frac{10}{100}\frac{1}{\sqrt{1+(\frac{(1000)(0.05)}{100})^2}} = 0.0894$ m/s,

$\phi(0.05) = \arctan[(1000)(0.05)/100] = 0.463r = 25.6\,\mathrm{deg}$

(iii) $\mathrm{M}(0.005) = \frac{10}{100}\frac{1}{\sqrt{1+(\frac{(1000)(0.005)}{100})^2}} = 0.0999$ m/s,

$\phi(0.005) = \arctan[(1000)(0.005)/100] = 0.050r = 2.86\,\mathrm{deg}$

(d) The limiting value of the magnitude of the response amplitude $M(\Omega)$ when $\Omega \to 0$ is $A_o = f_a/b = 0.10$ m/s. The ratio $M(\Omega)/A_o$ is 0.1961 when $\Omega = 0.5$ rad/sec, 0.894 when $\Omega = 0.05$ rad/sec, and 0.999 when $\Omega = 0.005$ rad/sec.

(e) The low-frequency asymptote for $M(\Omega)$ is $A_o = f_a/b$, while the high-frequency asymptote is $f_a/m\Omega$. These asymptotes intersect at the break frequency $\Omega_{break} = b/m = 0.10$ rad/sec.

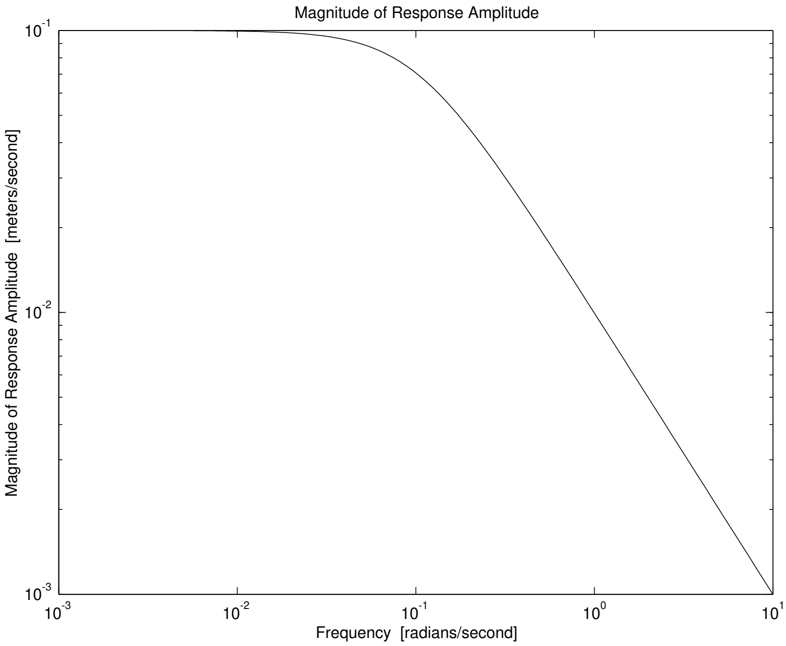

Figure 1: Magnitude of Response of Sliding Mass

(f) The Bode plots shown in Fig.1 and Fig.2 were obtained by running the following MATLAB script:

```
%Script for plotting Bode plots for 2.003 HW Assignment 8, Problem 1

Om = logspace(-3, 1, 200);
```

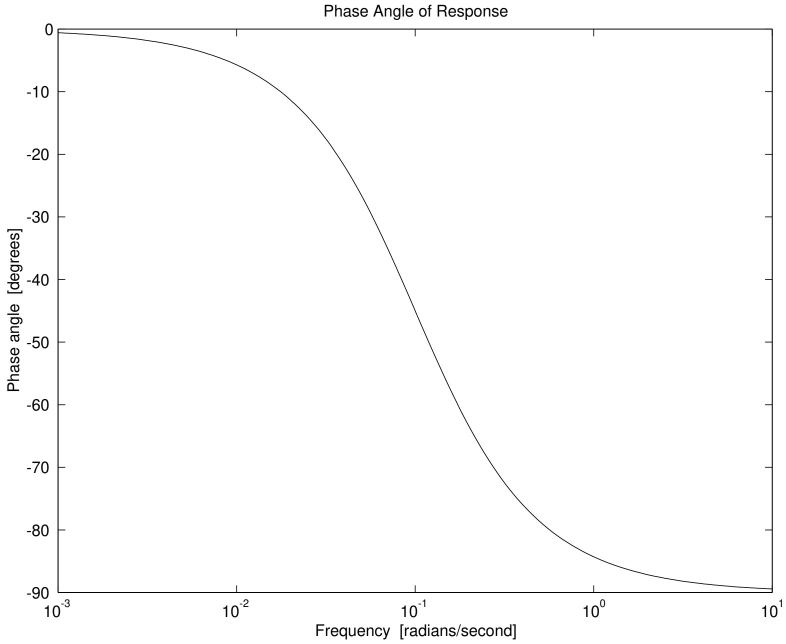

Figure 2: Phase Angle of Response of Sliding Mass

```
unity = ones(1,length(Om));
M = 0.1 ./ sqrt( unity + ( 10 .* Om).^2);
loglog(Om, M), title('Magnitude of Response Amplitude'),
xlabel('Frequency  [radians/second]'),
ylabel('Magnitude of Response Amplitude  [meters/second] '),
pause
phi = -57.3*atan(10 * Om);
semilogx(Om, phi), title('Phase Angle of Response'),
xlabel('Frequency  [radians/second]'),
ylabel('Phase angle  [degrees]')
```

2. **Rotating Inertia.** (25 points. 4 points each for part a, b, d and f, 6 points for part c, 3 points for part e) A rotor with moment of inertia I and angular velocity ω is subjected to an accelerating torque T and two retarding friction torques, each modeled linearly as $B\omega$.

(a) The equation of motion is

$$I\frac{d\omega}{dt} + 2B\omega = T \qquad \text{or} \qquad \frac{d\omega}{dt} = -\frac{2B}{I}\omega + \frac{1}{I}T \tag{2}$$

(b) The steady-state angular velocity, when $T = 10$ Newton-meters, and $I = 0.001$ kg-m^2, and $B = 0.005$ N-m/r/s is

$$\omega_{ss} = \frac{T}{2B} = \frac{10}{2(0.005)} = 1000 \text{ r/s}$$

(c) Consider the input $T = T_a \sin \Omega t$ to be the imaginary part of the complex input $T_a \exp(i\Omega t)$ and look for a complex solution of the form $\omega = A\exp(i\Omega t)$. After substitution in (2) we find that this is indeed a solution , if

$$A = \frac{T_a}{2B + iI\Omega} = \frac{T_a}{2B} \frac{1}{\sqrt{1 + (\frac{I\Omega}{2B})^2}} \exp(i\phi)$$

where the phase angle ϕ is fixed by the relation

$$\tan \phi = -\frac{I\Omega}{2B}$$

The steady-state solution to the input $T = T_a \sin \Omega t$ is then the imaginary part of the complex solution $A\exp(i\Omega t)$, which is

$$\omega = \frac{T_a}{2B} \frac{1}{\sqrt{1 + (\frac{I\Omega}{2B})^2}} \sin(\Omega t + \phi)$$

The magnitude of the response amplitude is

$$M(\Omega) = \frac{T_a}{2B} \frac{1}{\sqrt{1 + (\frac{I\Omega}{2B})^2}}$$

and the phase angle is

$$\phi = -\arctan \frac{I\Omega}{2B}$$

For the case where $T_a = 10$ Newton-meters, the values of $M(\Omega)$ and $\phi(\Omega)$ for the frequencies $\Omega = 50$ rad/sec, $\Omega = 5$ rad/sec, and $\Omega = 0.5$ rad/sec are

(i) $\mathrm{M}(50) = \frac{10}{0.01} \frac{1}{\sqrt{1+(\frac{(0.001)(50)}{0.01})^2}} = 196.1$ r/s,
$\phi(50) = \arctan[(0.001)(50)/0.01] = 1.373r = 78.7\,\mathrm{deg}$

(ii) $\mathrm{M}(5) = \frac{10}{0.01} \frac{1}{\sqrt{1+(\frac{(0.001)(5)}{0.01})^2}} = 894$ r/s,
$\phi(5) = \arctan[(0.001)(5)/0.01] = 0.463r = 25.6\,\mathrm{deg}$

(iii) $\mathrm{M}(0.5) = \frac{10}{0.01} \frac{1}{\sqrt{1+(\frac{(0.001)(0.5)}{0.01})^2}} = 999$ r/s,
$\phi(0.5) = \arctan[(0.001)(0.5)/0.01] = 0.0500r = 2.86\,\mathrm{deg}$

(d) The limiting value of the magnitude of the response amplitude $M(\Omega)$ when $\Omega \to 0$ is $A_o = T_a/2B = 1000$ r/s. The ratio $M(\Omega)/A_o$ is 0.1961 when $\Omega = 0.5$ rad/sec, 0.894 when $\Omega = 0.05$ rad/sec, and 0.999 when $\Omega = 0.005$ rad/sec.

(e) The low-frequency asymptote for $M(\Omega)$ is $A_o = T_a/2B$, while the high-frequency asymptote is $T_a/I\Omega$. These asymptotes intersect at the break frequency $\Omega_{break} = 2B/I = 10$ rad/sec.

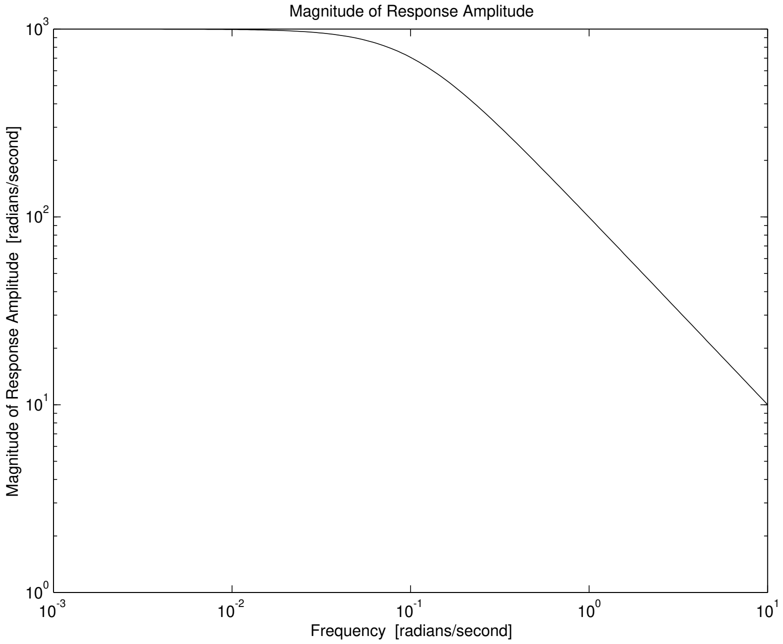

Figure 3: Magnitude of Response of Rotating Inertia

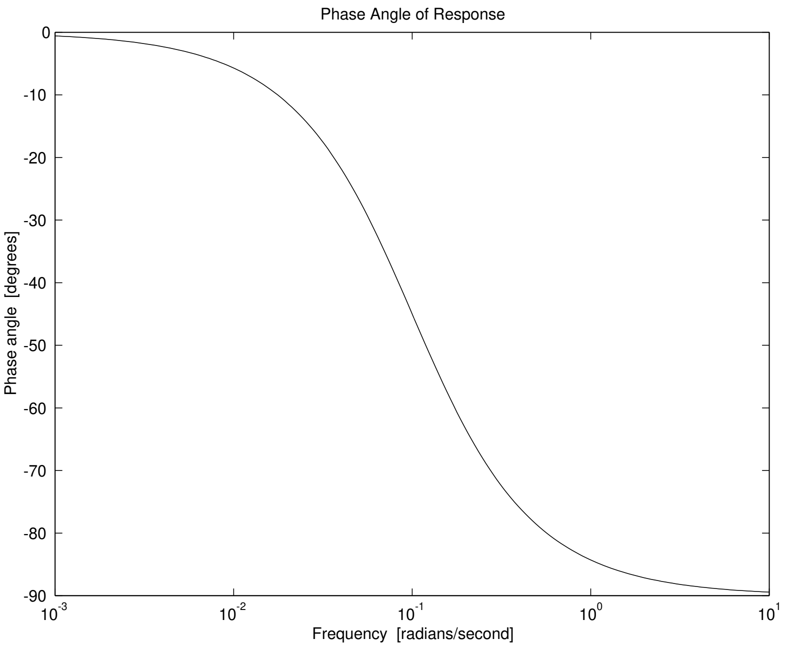

Figure 4: Phase Angle of Response of Rotating Inertia

(f) The Bode plots shown in Fig.3 and Fig.4 were obtaining by running a MATLAB script simolar to that used in Problem 1.

3. **Steady-state Sinusoidal Response of CD-Player.** (24 points. 6 points each for part a, b and d. 8 points for part c) The equation of motion for the CD-player is

$$I\frac{d\omega_m}{dt} + B_m\omega_m = T_m \qquad \text{or} \qquad \frac{d\omega_m}{dt} = -\frac{B_m}{I}\omega_m + \frac{1}{I}T$$

where: (i) $I = I_r$ when no disc is mounted on the spindle, or (ii) $I = I_r + I_d$ when a disc is mounted on the spindle.

(a) Using the result of Problem 2 with $B_m = 2B$, the magnitude $M(\Omega)$ of the angular velocity response amplitude and the phase angle $\phi(\Omega)$ for an input torque of $T_m = T_a \sin\Omega t$ are

$$M(\Omega) = \frac{T_a}{B_m}\frac{1}{\sqrt{1+(\frac{I\Omega}{B_m})^2}} \qquad \text{and} \qquad \phi(\Omega) = -\arctan\frac{I\Omega}{B_m}$$

The zero-frequency limit of the magnitude $M(\Omega)$ is

$$M(0) = \frac{T_a}{B_m}$$

Since no specific value of the input torque amplitude T_a was given, we will express the the magnitude of the response in terms of the ratio $M(\Omega)/M(0)$. For the three frequencies 1 rad/sec, 5 rad/sec, 15 rad/sec, the magnitude ratios and phase angles are:

(i) for no disc on the spindle ($I = I_r =$ 9e-6e-6 kg-m^2, $B_m =$ 9e-5 N-m/r/s),

$$\frac{M(1)}{M(0)} = \frac{1}{\sqrt{1+(\frac{(9\text{e-}6)(1)}{9\text{e-}5})^2}} = 0.995 \quad \text{and} \quad \phi = \arctan\frac{(9\text{e-}6)(1)}{9\text{e-}5} = 5.71\,\text{deg}$$

$$\frac{M(5)}{M(0)} = \frac{1}{\sqrt{1+(\frac{(9\text{e-}6)(5)}{9\text{e-}5})^2}} = 0.894 \quad \text{and} \quad \phi = \arctan\frac{(9\text{e-}6)(5)}{9\text{e-}5} = 26.6\,\text{deg}$$

$$\frac{M(15)}{M(0)} = \frac{1}{\sqrt{1+(\frac{(9\text{e-}6)(15)}{9\text{e-}5})^2}} = 0.555 \quad \text{and} \quad \phi = \arctan\frac{(9\text{e-}6)(15)}{9\text{e-}5} = 56.3\,\text{deg}$$

(ii) for a disc on the spindle ($I = I_r + I_d =$ 5.9e-5 kg-m^2, $B_m =$ 9e-5 N-m/r/s),

$$\frac{M(1)}{M(0)} = \frac{1}{\sqrt{1+(\frac{(5.9\text{e-}5)(1)}{9\text{e-}5})^2}} = 0.647 \quad \text{and} \quad \phi = \arctan\frac{(5.9\text{e-}5)(1)}{9\text{e-}5} = 33.2\,\text{deg}$$

$$\frac{M(5)}{M(0)} = \frac{1}{\sqrt{1+(\frac{(5.9\text{e-}5)(5)}{9\text{e-}5})^2}} = 0.1671 \quad \text{and} \quad \phi = \arctan\frac{(5.9\text{e-}5)(5)}{9\text{e-}5} = 73.0\,\text{deg}$$

$$\frac{M(15)}{M(0)} = \frac{1}{\sqrt{1+(\frac{(5.9\text{e-}5)(15)}{9\text{e-}5})^2}} = 0.0564 \quad \text{and} \quad \phi = \arctan\frac{(5.9\text{e-}5)(15)}{9\text{e-}5} = 84.2\,\text{deg}$$

(b) The low-frequency asymptote for $M(\Omega)/M(0)$ is unity, and the high frequency asymptote is $M(\Omega)/M(0) = B_m/I\Omega$. These two asymptotes intersect at the break frequency $\Omega_{break} = B_m/I$. The numerical values are:

(i) Ω_{break} = 9e-5/9e-6 = 10 rad/sec when no disc is mounted on the spindle.

(ii) Ω_{break} = 9e-5/5.9e-5 = 1.525 rad/sec when a disc is mounted on the spindle.

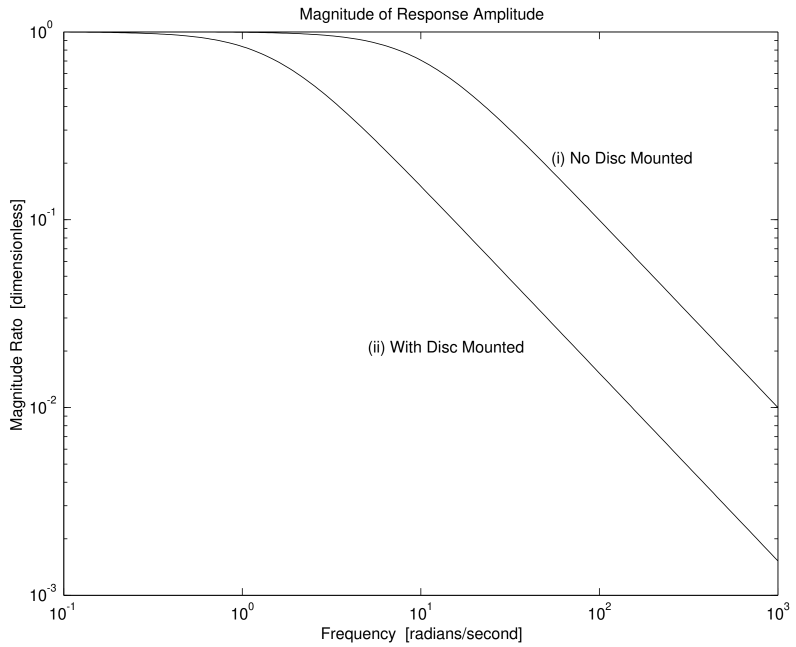

Figure 5: Magnitude Ratio for CD-Player

(c) Bode plots for Case (i) and Case (ii) are shown in Fig.5 for magnitude and in Fig.6 for phase angle. These plots were obtained by running MATLAB scripts similar to the one shown in Problem 1.

(d) The decay time-constant for the CD-player is $\tau = I/B_m$. The time for 98% of the transient to decay is $4\tau = 4I/B_m$. This time is a natural property of the system and is independent of the frequency Ω of the applied oscillating torque T_m. This time does, however, depend on the value of I which depends on whether a disc is mounted or not.

(i) With no disc mounted the time to reach steady state is 4(9e-6)/(9e-5) = 0.4 seconds.

(ii) With a disc mounted the time to reach steady state is 4(5.9e-5)/(9e-5) = 2.62 seconds.

4. **Vibration of an engine due to motion of a single piston.** (26 points. 6 points each for part a, b and d. 4 points each for part c and e.)

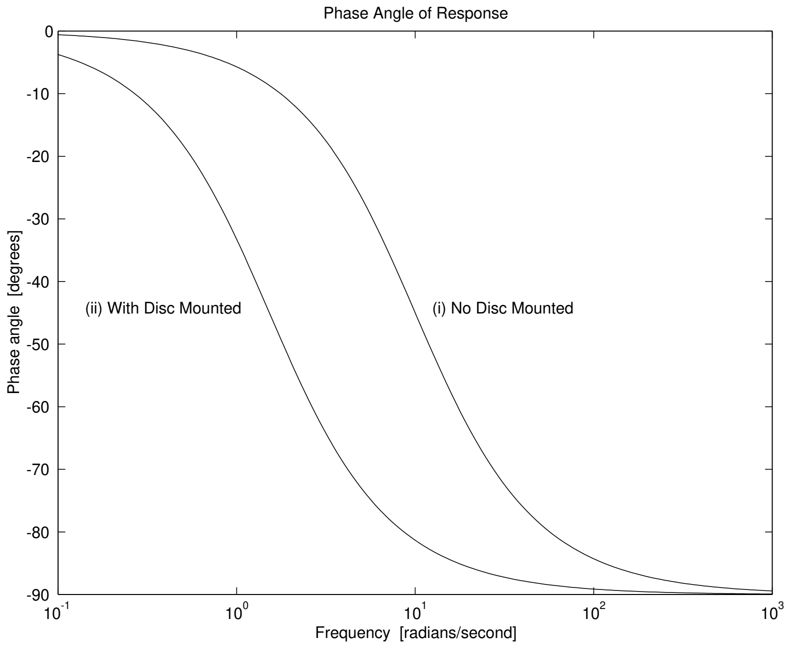

Figure 6: Phase Angle for CD-Player

(a) If mass m oscillates with displacement amplitude r at frequency Ω, the force required to accelerate it oscillates at frequency Ω with amplitude

$$f_a = mr\Omega^2 = \frac{w}{g}r\Omega^2 = \frac{2}{386}(1)(\frac{2\pi}{60}N)^2 \text{ pounds}$$

which gives $f_a = 127.8$ pounds (569 N) for $N = 1500$ rpm, $f_a = 511.4$ pounds (2275 N) for $N = 3000$ rpm, and $f_a = 2046$ pounds (9098 N) for $N = 6000$ rpm.

(b) An equation of motion for a mass M acted upon by a spring force $-ky$, a damping force $-b\frac{dy}{dt}$, and an applied force $f_a \sin \Omega t$ is

$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f_a \sin \Omega t$$

(c) Consider the input force to be the imaginary part of $f_a \exp(i\Omega t)$ and look for the steady-state solution as the imaginary part of the complex solution $y = A\exp(i\Omega t)$. The complex amplitude of the solution is

$$A = \frac{f_a}{k + ib\Omega - M\Omega^2}$$

The magnitude of A is

$$|A| = \frac{f_a}{\sqrt{(k - M\Omega^2)^2 + b^2\Omega^2}}$$

and the phase is

$$\phi = -\arctan\left(\frac{b\Omega}{k - M\Omega^2}\right)$$

(d) The magnitude of the amplitude of displacement response, when f_a has the value given in (a) above, with $Mg = 200$ pounds, k =18,000 pounds/inch, and $b = 2$ pounds/inch/sec is

$$|A| = \frac{5.68\text{e-}5N^2}{\sqrt{(18,000 - \frac{200}{386}(\frac{2\pi}{60}N)^2)^2 + (2\frac{2\pi}{60}N)^2}}$$

which gives $|A| = 0.0419$ inches (0.00106 meters) for $N = 1600$ rpm, $|A| = 0.331$ inches (0.00840 meters) for $N = 1800$ rpm, and $|A| = 0.0479$ inches (0.00122 meters) for $N = 2000$ rpm. Note strong effect of resonance at 1800 rpm.

(e) All of the above has to do with steady-state solutions. How long it takes to reach steady-state depends on how long it takes for the transient to decay, say to 2% of its initial value. This takes 4 decay time-constants. The decay time-constant is

$$\tau = \frac{1}{\zeta\omega_o} = 2\frac{M}{b} = 2\frac{200}{(386)(2)} = 0.518 \text{ seconds}$$

so the time to reach steady state is 2.07 seconds, independently of the engine speed N.