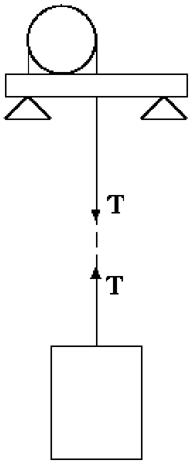
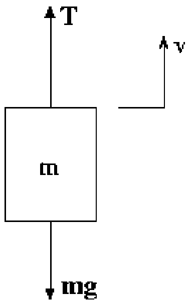
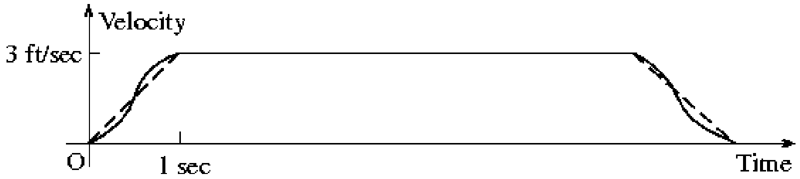
2.003 Fall 1999 Solution of Homework Assignment 1

1. Consider a free-body diagram of the elevator of mass m when it is accelerating upward with acceleration dv/dt. The equation of motion is

$$T - mg = m\frac{dv}{dt}$$







When the elevator is at rest, or traveling at constant velocity ($dv/dt = 0$), the tension T in the cable (and the load on the winch) would be simply the weight mg of the elevator and its cargo. Designing the winch structure on the basis of this load would be static design. Including the dynamic effect of accelerating the elevator results in $T = mg + mdv/dt$. To answer the question of whether dynamics matters, one must compare the magnitudes of the static load mg and the dynamic load $mg + mdv/dt$. The problem statement does not specify the explicit time behavior of dv/dt. It merely states that a total change of velocity of 3 feet/second is accomplished in one second. The sketch above shows two possible velocity histories. The slope of the velocity history is the acceleration dv/dt. With the dashed history the acceleration is uniform and has the magnitude

$$\frac{dv}{dt} = \frac{3\text{ft/sec}}{1\text{sec}} = (3)(0.3048) = 0.914\text{m/sec}^2$$

This time history has the smallest maximum acceleration but the sharp corners at the beginning and end of the one second acceleration are hard on the winch motor and are felt as jerks by the passengers. The time history suggested by the solid curve in the figure produces a smoother transition at the expense of a larger maximum acceleration. If the system is designed to accelearate according to the solid curve the maximum acceleration can be 50% to 100% larger than the uniform acceleration of the dashed history. In the latter case the ratio of the dynamic load to the static load would be

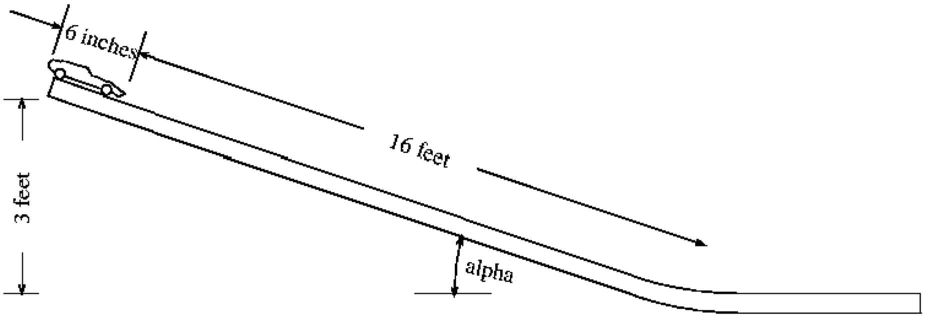
$$\frac{g + (dv/dt)_{max}}{g} = \frac{9.81 + 2(0.914)}{9.81} = 1.186$$

Here, dynamics does indeed matter! The dynamic load is 18.6% larger than the static load.

Engineers often make preliminary design calculations to determine ball-park estimates of size, weight, and power. In these 'back-of-the-envelope' calculations it is common to omit effects with contributions of the order of five to ten percent. However, these effects must be included in final design calculations, which have to be defended in safety investigations.

2. (a) The potential energy PE of a mass m elevated a distance h in the gravity field is PE $= mgh$. In SI units the 5 ounce mass is 5/16 pounds = (5/16)(0.4536 kg/pound) = 0.1418 kg. The 3 foot elevation h = 3(0.3048 meters/foot) = 0.914 meters, and the acceleration of gravity is 9.81 m/s^2. The energy available to move the racecar is

$$mgh = (0.1418)(9.81)(0.914) = 1.271\text{kg } (\text{m/s})^2 = 1.271\text{newton meters} = 1.271\text{Joules}$$



(b) If the center of mass is at the front of the car, its elevation is $h = 0.914$m and the available potential energy is as calculated in (a). When the car is rolling on the level portion of the track this energy has been transformed into kinetic energy KE= $\frac{1}{2}mv^2$, so the maximum velocity is

$$v = \sqrt{2\text{PE}/m} = \sqrt{2gh} = \sqrt{2(9.81)(0.914)} = 4.24 \text{ m/s}$$

(c) If the center of mass is at the rear of the car its elevation is

$$h = 0.914 + \frac{6}{12}(0.3048)\sin(\text{alpha}) = 0.914 + \frac{6}{12}(0.3048)(\frac{3}{16}) = 0.943 \text{ m}$$

and the maximum velocity on the level portion of the track is

$$v = \sqrt{2(9.81)(0,943)} = 4.30 \text{ m/s}$$

a 1.4 % increase in speed over the case where the mass center is at the front of the car.

(d) The energy relations are silent regarding the time taken to effect the energy transformation. To introduce the time variable it is necessary to consider the equation of motion of the car. On the sloping section of track the force which accelerates the mass along the track is the component of the weight parallel to the track: $mg\sin(\text{alpha})$. With no friction, this is the only force, and the equation of motion is

$$mg\sin(\text{alpha}) = m\frac{dv}{dt}$$

Integrating this equation from the initial condition $v = 0$ at $t = 0$ yields

$$v = gt\sin(\text{alpha})$$

Let s represent distance down the track from the starting point, so that $ds/dt = v$. Then integrating the equation

$$\frac{ds}{dt} = gt\sin(\text{alpha})$$

from $s = 0$ at $t = 0$ to $s = L$ at time $t = T$ yields

$$L = \frac{1}{2}gT^2\sin(\text{alpha})$$

from which it follows that

$$T = \sqrt{\frac{2L}{g\sin(\text{alpha})}}$$

The time T to reach the bottom of the inclined track in case (c) is obtained by substituting $L = 16.5$ feet $= (16.5)(0.3048)$m $= 5.09$m and $\sin(\text{alpha}) = 3/16$ to get

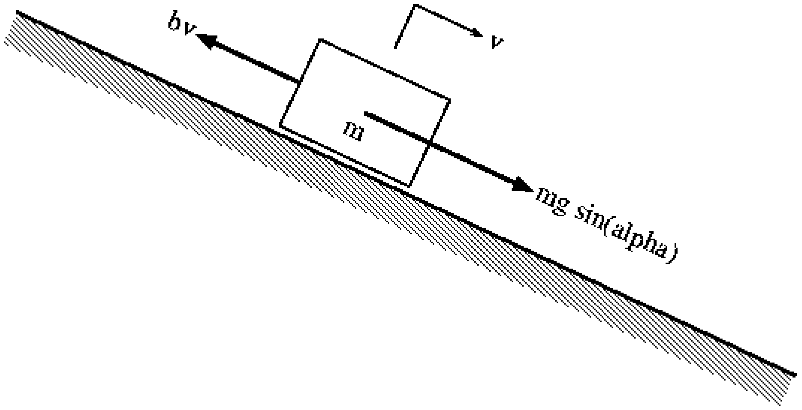
$$T = \sqrt{\frac{2(5.09)}{9.81(3/16)}} = 2.35 \text{ sec}$$

3. At the limiting velocity v_{ss}, the retarding force bv just equals the component of the weight parallel to the track, $mg\sin(\text{alpha})$, so

$$b = \frac{mg\sin(\text{alpha})}{v_{ss}}$$

If $v_{ss} = 2(4.30) = 8.60$ m/s, then

$$b = \frac{(0.1418)(9.81)(3/16)}{8.60} = 0.0303 \text{ kg/sec}$$



4. Consider the equation of motion of a racecar with mass m on an incline of angle (alpha), acted on by the weight component parallel to the track, and by a viscous retarding force bv, as shown in the Figure.

$$m\frac{dv}{dt} = mg\sin(\text{alpha}) - bv$$

or, after dividing through by m, and rearranging

$$\frac{dv}{dt} + \frac{b}{m} = g\sin(\text{alpha})$$

This is the equation treated in the MATLAB scripts 'car.m' and 'car_visc.m'.

For part (a) use the data $m = 0.1418$ kg, (alpha) = inv sin 3/16 = 10.81 degrees, $b = 0.0303$ kg/sec, and $T = 2.35$ sec to get the plot for Part (a).

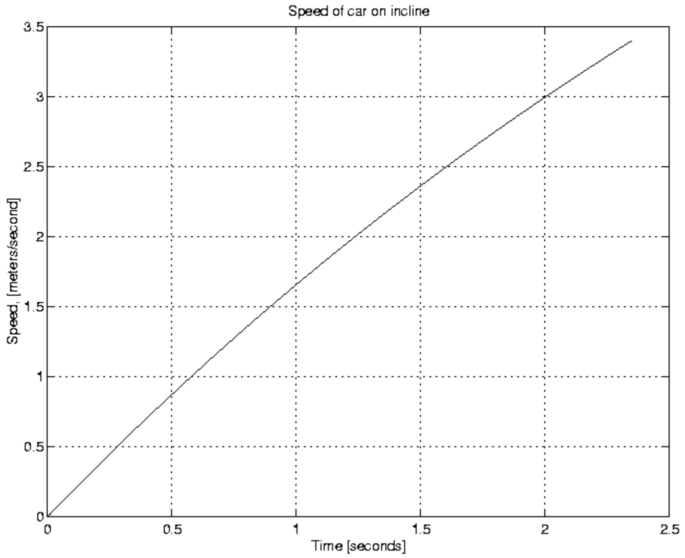


Figure 1: Plot for Part (a)

At the end of 2.35 seconds the speed of the car is 3.40 m/s. To indicate the effect of increasing the weight, you can run the program again with all the same data, except with the mass increased to 0.1701 kg (6 ounces) to get a final velocity of 3.53 m/s at the end of 2.35 seconds.

(b) The limiting speed or 'terminal velocity' established in Problem 3 is $v_{ss} = 8.60$ m/s. The speed which is 99.9 % of this is 8.591 m/s. To determine the time it takes to reach this speed, fix the values of $m = 0.1418$ kg, (alpha) = 10.81 degrees, and $b = 0.0303$ kg/sec, and run the program for a sequence of time intervals, iterating toward a final velocity of 8.591 m/s. For example,

Try T = 10 secs, get V = 7.59 m/s

Try T = 20 secs, get V = 8.49 m/s

Try T = 30 secs, get V = 8.596 m/s

Try T = 29 secs, get V = 8.593 m/s

Try T = 28 secs, get V = 8.589 m/s

Try T = 28.5 secs, get V = 8.591 m/s

The time history of velocity in the 28.5 second interval is displayed in the plot for Part (b).

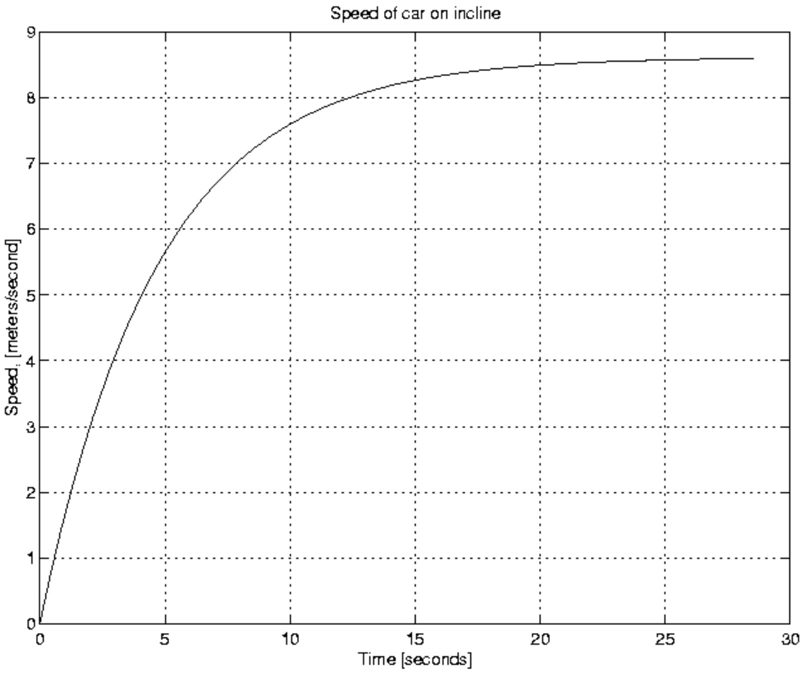


Figure 2: Plot for Part (b)

The previous result is not very accurate because of round-off error. If we test the MATLAB progams with the given data, we find that for the very large times, T = 70 to T = 100, the final velocity is essentially constant at $v = 8.61045$ m/s; *i.e.,* the program believes that the terminal velocity is not 8.60 m/s, but 8.61045 m/s. If we then iterate to a final velocity of 8.602 m/s, which is 99.9% of 8.61045 m/s, we find that the time required is T = 32.5 sec, which is 14% larger than our previous result of 28.5 sec.