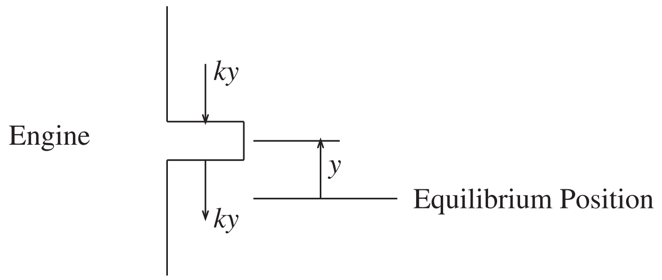
2.003 Fall 1999

QUIZ SOLUTION

1. Engine mounted on four elastomeric mounts, each of which consists of two elastomeric pads. Each pad has stiffness k and damping coefficient b.

(a{i}) When the engine is displaced a distance y from its equilibrium position both pads in each mount experience a deflection y and they react with restoring forces ky as shown in the sketch. The total restoring force at one mount is $2ky$, so the effective stiffness at each mount is $2k$.

(a{ii}) A parallel argument with velocity v in place of displacement y, and damping coefficient b in place of stiffness k, leads to the conclusion that the total force opposing velocity at each mount is $2bv$, so the effective damping coefficient at each mount is $2b$



(b{i}) When the engine is displaced a distance y from its equilibrium position each mount reacts with a restoring force $2ky$, and the total restoring force acting on the engine is $8ky$, so the effective stiffness of all four mounts is $8k$.

(b{ii}) A parallel argument with velocity v in place of displacement y, and damping coefficient b in place of stiffness k, leads to the conclusion that the total force opposing velocity is $8bv$, so the effective damping coefficient at each mount is $8b$.

(c) Take the engine to be a rigid body with mass m, supported by a suspension with effective stiffness $8k$ and effective damping coefficient $8b$, and acted upon by an applied force f_a. The fundamental requirements are:

Geometric Compatibility $\qquad v = dy/dt$

Constitutive Equations $\qquad f_m = mdv/dt, \qquad f_k = 8ky, \qquad$ and $\qquad f_b = 8bv$

Force Balance $\qquad f_m = f_a - f_k - f_b$

These can be combined into the state equations for a state-determined system with y and v as state variables and with f_a as input, and y as output. In standard matrix form, the equations are:

$$\frac{d}{dt}\left\{\begin{array}{c} y \\ v \end{array}\right\} = \left[\begin{array}{cc} 0 & 1 \\ -\frac{8k}{m} & -\frac{8b}{m} \end{array}\right]\left\{\begin{array}{c} y \\ v \end{array}\right\} + \left\{\begin{array}{c} 0 \\ \frac{1}{m} \end{array}\right\} f_a$$

with output

$$y = \{\ 1 \quad 0\ \}\left\{\begin{array}{c} y \\ v \end{array}\right\}$$

(d) If the engine weight is W, then its mass is $m = W/g$. The damping ratio ζ and undamped natural frequency ω_o can be obtained from the equations

$$2\zeta\omega_o = \frac{b_{eff}}{m} \qquad \text{and} \qquad \omega_o^2 = \frac{k_{eff}}{m}$$

The result for ω_o, (d{ii}), is

$$\omega_o = \sqrt{\frac{8kg}{W}}$$

and the result for ζ, (d{i}), is

$$\zeta = \frac{b_{eff}}{2\omega_o m} = \frac{1}{2\omega_o}\frac{8bg}{W} = \frac{1}{2}\sqrt{\frac{W}{8kg}}\frac{8bg}{W} = b\sqrt{\frac{2g}{kW}}$$

2. A 150-pound diver initiates an oscillation of a diving board. The plot of the displacement response exhibits a damped natural period of $T_d = 0.5$ seconds, and a succession of peak amplitudes of approximately 5.85 in, 3.10 in, 1.65 in, and 0.85 in.

(a{i}) The natural logarithm of the ratio of the magnitudes of any two *succeeding* peak amplitudes provides an estimate of the LDR. The individual ratios vary from 0.515 to 0.532. The average of all three is 0.526 and the corresponding LDR is -0.642. The damping ratio ζ is obtained from the equation

$$\zeta^2 = \frac{LDR^2}{\pi^2 + LDR^2} = 0.0401$$

The result is $\zeta = 0.200$.

(a{ii}) The damped natural frequency ω_d is obtained from the measured damped natural period T_d.

$$\omega_d = \frac{2\pi}{T_d} = \frac{2\pi}{0.5} = 12.57 \text{ radians/second}$$

(a{iii}) The undamped natural frequency ω_o is

$$\omega_o = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{12.57}{\sqrt{1-(0.200)^2}} = 12.83 \text{ radians/second}$$

(b{i}) When a mass m is placed on a board with effective stiffness k in the gravity field, the static deflection Δh is

$$\Delta h = \frac{mg}{k} = \frac{g}{\omega_o^2} = \frac{386}{(12.83)^2} = 2.35 \text{ inches} \quad \text{or} \quad \frac{9.81}{(12.83)^2} = 0.0596 \text{ meters}$$

(b{ii}) The effective stiffness k is

$$k = \frac{W}{\Delta h} = \frac{150}{2.35} = 64.0 \text{ pounds/inch} \quad \text{or} \quad \frac{(150)(4.448)}{0.0596} = 11,200 \text{ Newtons/meter}$$

(b{iii}) The effective damping of the board is

$$b = 2\zeta\omega_o m = 2\zeta\omega_o\frac{W}{g} = 2(0.200)(12.83)\frac{150}{386} = 1.995 \text{ pounds/inch/second}$$

or, in SI units,

$$b = 2\zeta\omega_o\frac{W}{g} = 2(0.200)(12.83)\frac{(150)(4.448)}{9.81} = 349 \text{ Newtons/meter/second}$$

3. This is a drill on first-order and second-order responses. Six pairs of responses (unit step response and unit impulse response) are displayed in Figs.1-6. Fig.1 is for a first-order system with a decay time-constant of 0.5 seconds. Figs. 2-5 are for second-order systems with decreasing damping ratios ζ. Fig. 2 is for a value of ζ considerably larger than unity (the responses are very similar to first-order responses, except that the impulse response clearly starts from zero). Fig.3 is for a value of ζ only slightly larger than unity. Fig.4 is for a value of ζ less than unity because of the oscillations. The amount of overshoot is however very small, indicating a ζ-value in the neighborhood of 0.7 to 0.8. finally there are sizeable oscillations in Fig. 5 indicating a ζ value in the neighborhoood of 0.20 to 0.4. The responses in Fig.6 differ from the preceding four second-order responses because of different initial consitions. However, after the first overshoot, the responses look like typical second-order responses. The undershoot in the impulse response is followed by a mild oscillation which appears to have a smaller damping ratio than Fig.4, but a larger damping ratio than Fig.5.

The four eigenvalue plots show the locations of the roots λ obtained by inserting $y = Ce^{\lambda t}$ in the differential equations for free vibration of certain first and second-order differential equations. In Plot 1, there is a single real root, $\lambda = -2$ per second. In Plot 2 there are a pair of real roots close to the value $\lambda = -5$ per second. In Plot 3 there are a pair of complex roots lying on the circle of radius $\omega_o = 5$ radians/second at an angle close to 45 degrees, which indicates a damping ratio ζ close to $\zeta = 0.707$. In Plot 4 there are a pair of complex roots lying on the circle of radius $\omega_o = 5$ radians/second with fairly light damping (the damped natural frequency is only slightly smaller than the undamped natural frequency). The real part of $\lambda = -\zeta\omega_o$ is about -1.5 per second, so ζ is about $1.5/5 = 0.3$.

The five differential equations listed in part (b) divide into two categories. The first two represent first-order systems and the last three represent second-order systems. The decay time-constant in Eq.(1) is 0.5 seconds. The decay time-constant in Eq.(2) is 2.0 seconds. All three of the second-order equations represent systems with an undamped natural frequency of 5.0 radians/second. Eq.(3) represents a system with damping ratio $\zeta = 1.01$, while Eqs.(4) and (5) represent systems with damping ratios of 0.7 and 0.3, respectively.

(a) The match-up of eigenvalues with transient response plots is:

Eigenvalue Plot	Transient Response Plot
1	Fig.1
2	Fig.3
3	Fig.4
4	Fig.5

(b) The match-up of eigenvalues with differential equations is:

Eigenvalue Plot	Equation Number
1	Eq.1
2	Eq.3
3	Eq.4
4	Eq.5

(c) The state-determined system equivalent to Eq.(1) is

$$\frac{dx}{dt} = -2.0x + 2.0u \qquad \text{with output} \qquad y = x$$

and the state-determined system equivalent to Eq.(3) is

$$\frac{d}{dt}\left\{ \begin{array}{c} x \\ v \end{array} \right\} = \left[\begin{array}{cc} 0 & 1 \\ -25 & -10.1 \end{array} \right] \left\{ \begin{array}{c} x \\ v \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ 25 \end{array} \right\} u$$

with output

$$y = \{\, 1 \quad 0 \,\} \left\{ \begin{array}{c} x \\ v \end{array} \right\}$$