2.010 Fall 2000

Quiz 2 Solution

Problem 1 Torsional Vibration

1. By inspection there are two modes of vibration, a "rigid-body" mode in which engine, shaft and load rotate at the same speed and a "first mode" in which the engine and load rotate in opposite directions, twisting the shaft. Equations of motion are

$$\frac{d}{dt}\left(I_m\dot{\theta}_m\right)=\tau-K\left(\theta_m-\theta\right) \text{ and } \frac{d}{dt}\left(I\dot{\theta}\right)=K\left(\theta_m-\theta\right) \qquad (1)$$

Rewrite in matrix form

$$\begin{bmatrix} I & 0 \\ 0 & I_m \end{bmatrix}\begin{Bmatrix} \ddot{\theta} \\ \ddot{\theta}_m \end{Bmatrix}+\begin{bmatrix} K & -K \\ -K & K \end{bmatrix}\begin{Bmatrix} \theta \\ \theta_m \end{Bmatrix}=\begin{Bmatrix} 0 \\ \tau \end{Bmatrix}$$

To find natural modes of vibration, assume $\tau=0$. As the system has no damping, the unforced responses will be sinusoids of the form

$$\begin{Bmatrix} \theta \\ \theta_m \end{Bmatrix}=\begin{Bmatrix} a \\ a_m \end{Bmatrix}\sin(\omega t)$$

Differentiating twice

$$\begin{Bmatrix} \ddot{\theta} \\ \ddot{\theta}_m \end{Bmatrix}=-\omega^2\begin{Bmatrix} a \\ a_m \end{Bmatrix}\sin(\omega t)$$

Substituting

$$\begin{bmatrix} K-I\omega^2 & -K \\ -K & K-I_m\omega^2 \end{bmatrix}\begin{Bmatrix} a \\ a_m \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (2)$$

Non-trivial solutions require the determinant to be zero which yields the characteristic equation.

$$I_m I\omega^4-K\left(I_m+I\right)\omega^2=0$$

Solutions are $\omega_0^2=0$ (the "rigid body" mode) and $\omega_1^2=K\dfrac{I_m+I}{I_m I}$ (the "first mode").

These are the required algebraic equations.

You weren't asked to find the mode shapes, but they may be obtained by substituting the natural frequencies into (2). For $\omega_0^2 = 0$, $a = a_m$ and for $\omega_1^2 = K\dfrac{I_m + I}{I_m I}$, $a = -\dfrac{I_m}{I}a_m$.

Rigid-body mode: $\begin{Bmatrix}1\\1\end{Bmatrix}$ First mode: $\begin{Bmatrix}1\\-I_m/I\end{Bmatrix}$

2. The frequency response function is the product of two terms due to the rigid-body and first modes. In the rigid-body mode all of the system inertia, $I_m + I$, is accelerated in unison. The frequency response relating input motor torque to output load angular velocity exhibits no resonance but a phase of –90° at all frequencies and a magnitude that declines at –20 dB/decade of increasing frequency. At a frequency of 1 radian/second, $\left\|\dot{\theta}/\tau\right\|_{\Omega=1} = 1/3$ which corresponds to –9.5 dB.

The first mode exhibits an undamped (infinite) resonant peak at

$$\omega_1 = \sqrt{K\frac{I_m + I}{I_m I}} = \sqrt{600\frac{1+2}{2}} = 30 \text{ radian/second}$$

Below that frequency the phase is zero while the magnitude due to this mode asymptotically approaches unity (0 dB). Above that frequency, the phase abruptly switches to –180° while the magnitude due to this mode declines at a rate asymptotically approaching –40 dB/decade.

With this information an accurate sketch of the Bode plots can be made. The figure on the following page shows Bode plots produced by MATLAB.

If the above reasoning was not obvious, an alternative approach is to re-write the differential equations (1) in the Laplace domain to find the transfer function as follows.

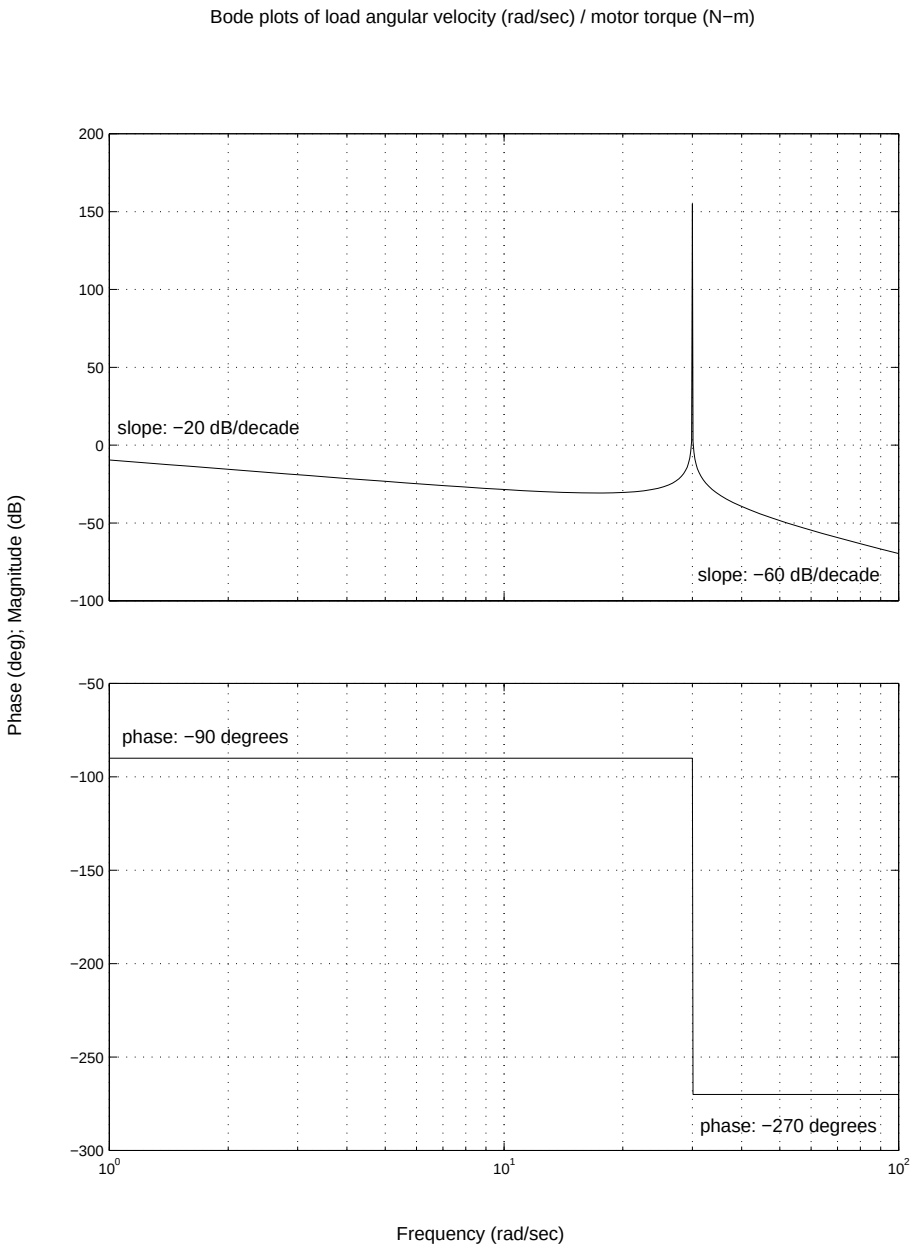
$$\left(I_m s^2 + K\right)\theta_m = \tau + K\theta \text{ and } \left(Is^2 + K\right)\theta = K\theta_m$$

Eliminate θ_m

$$\frac{\left(I_m s^2 + K\right)\left(Is^2 + K\right)}{K}\theta = \tau + K\theta$$

$$\left(I_m I s^4 + K\left(I_m + I\right)s^2\right)\theta = K\tau$$

$$\frac{\theta(s)}{\tau(s)} = \frac{K}{I_m I s^4 + K\left(I_m + I\right)s^2} = \left(\frac{1}{\left(I_m + I\right)s^2}\right)\left(\frac{K\dfrac{I_m + I}{I_m I}}{s^2 + K\dfrac{I_m + I}{I_m I}}\right) = \left(\frac{1}{\left(I_m + I\right)s^2}\right)\left(\frac{\omega_1^2}{s^2 + \omega_1^2}\right)$$



That's for output position; for output velocity multiply by *s*.

$$\frac{\dot{\theta}(\Omega)}{\tau(\Omega)} = \left(\frac{1}{(I_m + I)s} \right) \left(\frac{\omega_1^2}{s^2 + \omega_1^2} \right)$$

To find the frequency response function substitute $s = j\Omega$.

$$\frac{\dot{\theta}(\Omega)}{\tau(\Omega)} = \left(\frac{1}{(I_m + I)j\Omega} \right) \left(\frac{\omega_1^2}{-\Omega^2 + \omega_1^2} \right)$$

This is the product of two terms due to the rigid-body and first modes as described above.

3. The sensor is at a node.

That's six words; To be more long-winded, in the higher-frequency (first) mode of vibration the engine and load inertias move in opposite directions, hence there must be a point on the shaft that does not move – it is called a node. In general, the unforced response of this system is a weighted sum of the two modes of vibration. A sensor placed at the point on the shaft where the node of the first mode is located will only "see" motion due to the lower-frequency (rigid-body) mode.

4. In the first mode of vibration, the part of the system to the left (or right) of the node behaves as if the node were attached to the inertial reference frame. If the node is a distance l_m from the engine, the stiffness of the shaft to the left of the node is

$$K_{left} = K\frac{l}{l_m}$$

and the undamped natural frequency of the first mode is

$$\omega_1^2 = K\frac{l}{l_m}\frac{1}{I_m}$$

Equate this to the earlier expression for the first mode natural frequency and solve for l_m.

$$K\frac{l}{l_m I_m} = K\frac{I_m + I}{I_m I}$$

$$l_m = l\frac{I}{I_m + I} = 3\times\frac{1}{1+2} \text{ meters}$$

The sensor is located 1 meter from the engine.

Problem 2 Hydraulic Drive

1. Because the observed behavior is a first-order linear dynamic relation between servovalve input current, i, and rotary inertia angular velocity, ω, we may reasonably assume that (a) all model elements are linear and (b) the system dynamic behavior is due to one independent inertial energy storage element retarded by some power-dissipation element(s) and (c) there is no significant elasticity anywhere in the system. (Any elasticity coupled to this inertia, whether of mechanical or fluid origin, would result in at least two independent energy-storage elements and at least a second-order model.) As a result, this system has only one degree of freedom – the rotor and pinion have the same angular velocity which is algebraically related to the common translational velocity, v, of the rack, rod and piston and also to the fluid flow rate, Q.

An easy way to organize model development is to follow the path of power transmission. In this system it is

hydraulic servovalve ⇔ piston & cylinder ⇔ rack & pinion ⇔ rotary inertia

The output of interest is the angular velocity of the rotary inertia. Write constitutive equations for elements working back from that point.

$$I\dot{\omega} = \tau$$

where *I* is rotary inertia and τ is accelerating torque. The rotor is accelerated by the rack & pinion and possibly opposed by friction.

$$\tau = \tau_{rack} - B\omega$$

where *B* is a rotary damping coefficient and the subscript is self-explanatory. For the rack and pinion we know that $x = R\theta$. Differentiate to find a velocity relation $\dot{x} = v = R\dot{\theta} = R\omega$. Power losses in the rack and pinion may be neglected or lumped with rotary damping on the pinion side and/or translational damping on the rack side. With no power loss in the rack and pinion, power in = power out: $\tau_{rack}\omega = f_{rack}v$ where f_{rack} is the force on the pinion from the rack, hence $\tau_{rack} = Rf_{rack}$.

The force on the pinion from the rack is equal to the force on the piston, *f*, less the force required to accelerate any translational inertia of the rack, rod and piston and the force required to overcome translational friction (likely due to the seals between rod, piston and cylinder).

$$f_{rack} = f - \left(m_{piston} + m_{rod} + m_{rack}\right)\dot{v} - bv$$

where *m* denotes translational mass with self-explanatory subscripts and *b* is a translational damping coefficient. The force on the piston is proportional to the pressure drop across it $f = AP$. Pressure drop, *P*, is determined by the servovalve. Rearranging the valve characteristic,

$$P = \left(\frac{P_{\max}}{i_{\max}}\right)i - \left(\frac{P_{\max}}{Q_{\max}}\right)Q$$

The fluid flow rate is determined by the speed of the piston, $Q = Av$. Successive substitutions yield a first-order differential equation:

$$I\dot{\omega} = -B\omega + R\left(f - \left(m_{piston} + m_{rod} + m_{rack}\right)R\dot{\omega} - bR\omega\right)$$

$$\left(I + \left(m_{piston} + m_{rod} + m_{rack}\right)R^2\right)\dot{\omega} + \left(B + bR^2\right)\omega = RA\left(\left(\frac{P_{\max}}{i_{\max}}\right)i - \left(\frac{P_{\max}}{Q_{\max}}\right)AR\omega\right)$$

$$\left(I + \left(m_{piston} + m_{rod} + m_{rack}\right)R^2\right)\dot{\omega} + \left(B + \left(b + \left(\frac{P_{\max}}{Q_{\max}}\right)A^2\right)R^2\right)\omega = RA\left(\frac{P_{\max}}{i_{\max}}\right)i$$

To reduce the clutter define $I_{total} = I + \left(m_{piston} + m_{rod} + m_{rack}\right)R^2$ and

$B_{total} = B + \left(b + \left(\frac{P_{\max}}{Q_{\max}}\right)A^2\right)R^2$ and rewrite the differential equation.

$$I_{total}\dot{\omega} + B_{total}\omega = \left(\frac{RAP_{\max}}{i_{\max}}\right)i$$

In the Laplace domain

$$\left(I_{total}s + B_{total}\right)\omega(s) = \left(\frac{RAP_{\max}}{i_{\max}}\right)i(s)$$

Rearrange

$$\frac{\omega(s)}{i(s)} = \left(\frac{RAP_{\max}}{i_{\max}B_{total}}\right)\left(\frac{\frac{B_{total}}{I_{total}}}{s + \frac{B_{total}}{I_{total}}}\right)$$

The observed behavior is competently modeled by

$$\frac{\omega(s)}{i(s)} = K\frac{\lambda}{s + \lambda}$$

Equating terms yields the required algebraic equations.

$K = \frac{RAP_{\max}}{i_{\max}B_{total}}$ and $\lambda = \frac{B_{total}}{I_{total}}$ where B_{total} and I_{total} are defined above.

Given the problem statement one might further assume that each of *B*, *b*, m_{piston}, m_{rack} and m_{rod} are negligible. In that case $I_{total} = I$ and $B_{total} = \left(\frac{P_{\max}}{Q_{\max}}\right)A^2R^2$ and the algebraic expressions simplify to

$$K = \frac{Q_{\max}}{i_{\max}AR} \text{ and } \lambda = \frac{P_{\max}A^2R^2}{Q_{\max}I}$$

Note that the above analysis is not required for any of the remaining parts of this problem.

2. In the Laplace domain the controller and sensor equations are

$i(s) = G(1 + K_d s)(e_r(s) - e_s(s))$ and $e_s(s) = M\theta(s)$ with $M = 1$ volt/rad

Position is the time-integral of velocity so the physical system model is

$$\frac{\theta(s)}{i(s)} = \frac{K}{s}\frac{\lambda}{s+\lambda}$$

The open-loop transfer function is

$$\frac{\theta(s)}{\Delta e(s)} = G(1 + K_d s)\frac{K}{s}\frac{\lambda}{s+\lambda}$$

where $\Delta e = e_r - e_s$ hence the closed-loop transfer function is

$$\frac{\theta(s)}{e_r(s)} = \frac{G(1 + K_d s)\frac{K}{s}\frac{\lambda}{s+\lambda}}{1 + G(1 + K_d s)\frac{K}{s}\frac{\lambda}{s+\lambda}} = \frac{GK_d K\lambda(s + 1/K_d)}{s^2 + \lambda(1 + GK_d K)s + GK\lambda}$$

3. The closed-loop zero is defined by $s + 1/K_d = 0$ so a zero at –2 rad/sec requires

$K_d = 0.5$ sec/rad.

Equating the characteristic polynomial to a standard second-order form $s^2 + \lambda(1 + GK_d K)s + GK\lambda = s^2 + 2\zeta\omega_n + \omega_n^2$ an undamped natural frequency of 2 rad/sec requires $GK\lambda = 4$ and using $K = 1$ rad/sec/amp and $\lambda = 1$ rad/sec (and implicitly $M = 1$ volt/rad)

$G = 4$ amp/volt

The corresponding damping ratio is $\zeta = \frac{\lambda(1 + GK_d K)}{2\omega_n} = \frac{1 \times (1 + 4 \times 0.5 \times 1)}{2 \times 2} = 0.75$

4. Using the approximate differentiator the open loop transfer function is

$$\frac{\theta(s)}{\Delta e(s)} = G\left(1 + K_d s\frac{1/RC}{s + 1/RC}\right)\frac{K}{s}\frac{\lambda}{s+\lambda} = G\left(\frac{s(1 + K_d/RC) + 1/RC}{s + 1/RC}\right)\frac{K}{s}\frac{\lambda}{s+\lambda}$$

The effect of the approximate differentiation is to add a first-order pole to the open-loop transfer function and also slightly move the zero. The pole is at $\frac{1}{10^5 \times 10^{-7}} = 100$ rad/sec. The zero is defined by $s(1 + K_d/RC) + 1/RC = 0$ and its new location is at

$$z = -\frac{1/RC}{1 + K_d/RC} = -\frac{100}{1 + 100/2} = -1.96 \text{ rad/sec}$$

The open loop transfer function is the product of four terms hence the Bode plots are the sum of four terms.

(i) $G = 4$: $20\log_{10} G \cong 12$ dB, phase angle 0º

(ii) $\frac{s(1 + K_d/RC) + 1/RC}{s + 1/RC}$: The low frequency magnitude asymptote is unity or 0 dB, the low frequency phase asymptote is 0º. The zero is at 1.96 rad/sec. Above that frequency the magnitude increases at a rate that asymptotically approaches 20 dB/decade. At the zero break frequency the phase is 45º and above that frequency the phase asymptotically approaches 90º. The pole is at 100 rad/sec. Above that frequency the magnitude due to the pole decreases at a rate that asymptotically approaches –20 dB/decade, offsetting the increasing magnitude due to the zero, hence the high frequency magnitude is a constant equal to $1 + K_d/RC = 51$ $20\log_{10} 51 \cong 34$ dB. At the pole break frequency the phase due to the pole is –45º and above that frequency the phase due to the pole asymptotically approaches –90º offsetting the phase due to the zero, hence the high frequency phase asymptote is 0º.

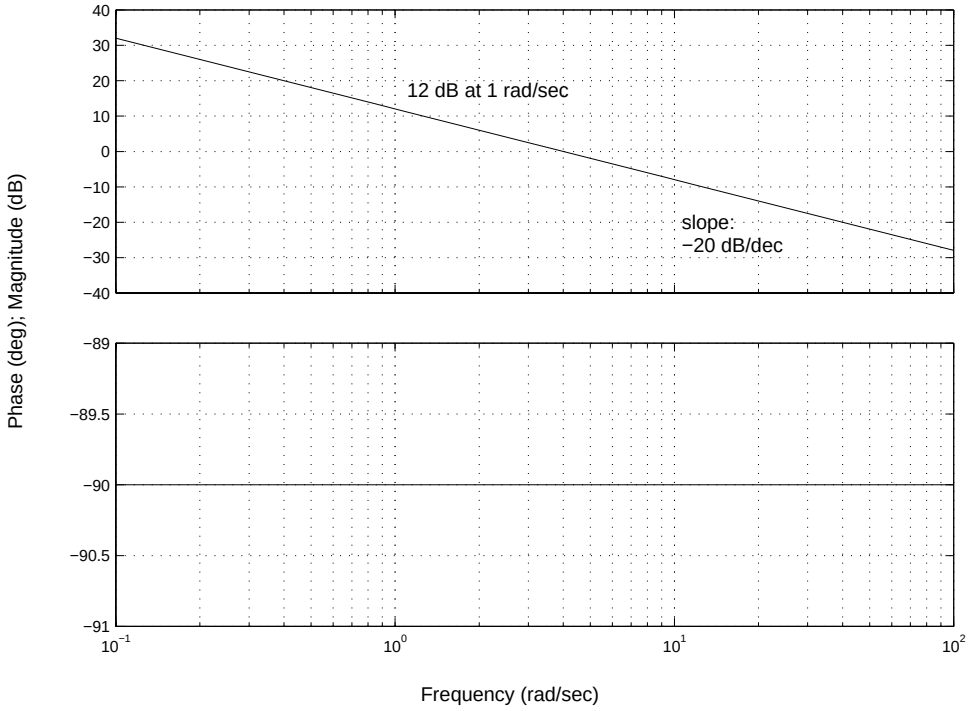
(iii) $\frac{K}{s}$: This is a pure integrator. The magnitude decreases uniformly with increasing frequency at -20 dB/decade, crossing 0 dB at 1 rad/sec. The phase is –90º at all frequencies.

(iv) $\frac{\lambda}{s + \lambda}$: The low frequency magnitude asymptote is unity or 0 dB, the low frequency phase asymptote is 0º. The pole is at 1 rad/sec. Above that frequency the magnitude decreases at a rate that asymptotically approaches –20 dB/decade. At the pole break frequency the phase is –45º and above that frequency the phase asymptotically approaches –90º.

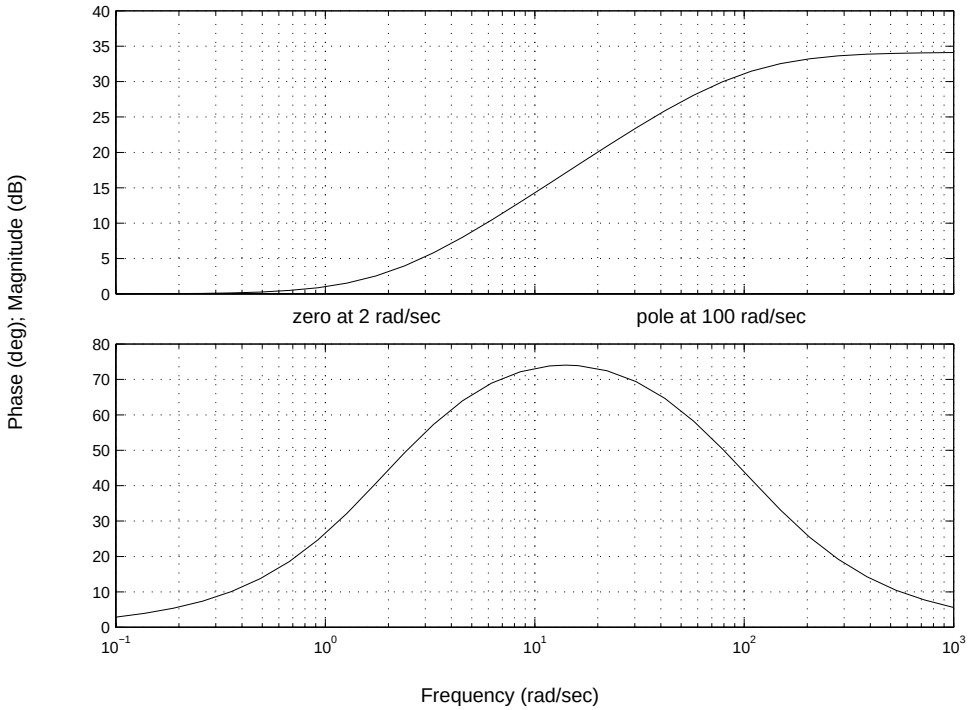
The combination of all four terms has a low frequency magnitude asymptote decreasing at –20 dB/decade, low frequency phase asymptote at –90º, high frequency magnitude asymptote decreasing at –40 dB/decade, high frequency phase asymptote at –180º.

The following figures show Bode plots produced by MATLAB of (i) GK/s (ii) $\frac{s(1 + K_d/RC) + 1/RC}{s + 1/RC}$ (iii) $\frac{\lambda}{s + \lambda}$ and (iv) the open-loop transfer function.

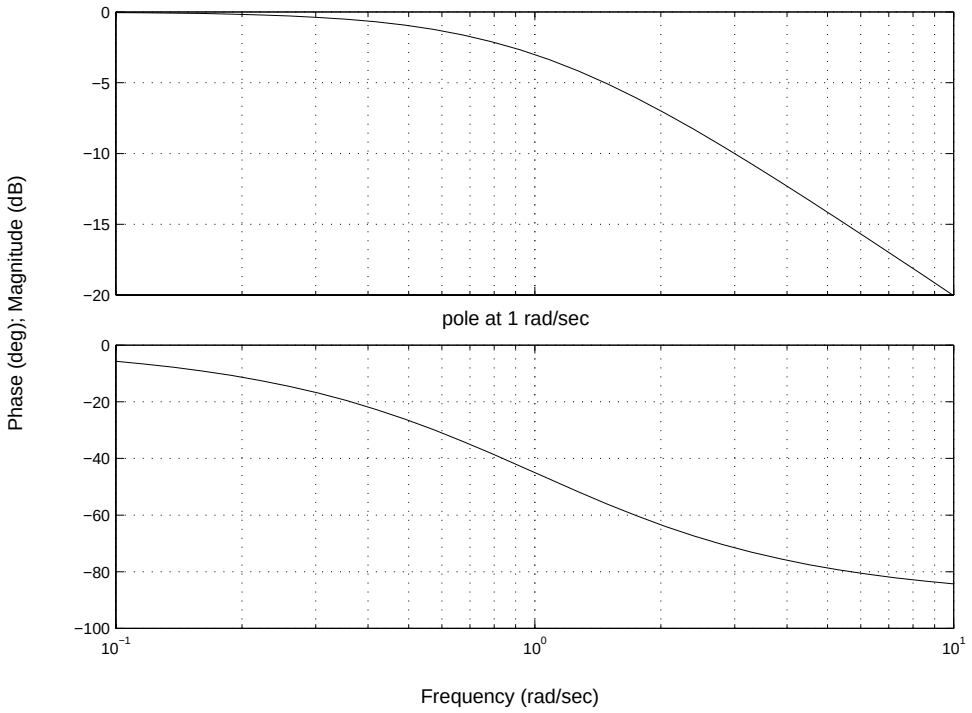
Bode plots for GK/s



Bode plots for approximate PD controller



Bode plots for angular velocity / servovalve current



Bode plots for open−loop transfer function

