MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Quantum Information Science I

October 21, 2010

Midterm Examination Solution

1. **Single qubit operations**.

   (a) Recall that $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$. Give $SXS^\dagger$, $SYS^\dagger$, and $SZS^\dagger$, for X, Y, and Z being the usual Pauli gates. (2 points each)
   Answer:

$$SXS^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y$$

$$SYS^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = -X$$

$$SZS^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z$$

   (b) Construct the single qubit gate $U = e^{i\pi/4}\exp(-i\pi Y/4)$ from S, H (Hadamard), and Pauli gates. (14 points)
   Answer:

$$SHSHS^3 = SHSHS^\dagger = e^{i\pi/4}\exp(-i\pi Y/4)$$

2. **Entanglement distillation by non-projective measurement**. Suppose Alice and Bob share the two-qubit state $|\psi_{AB}\rangle = (\sqrt{3}|00\rangle + |11\rangle)/2$. Recall that a quantum measurement is specified by a set of operators $\{M_0, M_1\}$ such that $\sum_k M_k^\dagger M_k = I$.

   (a) Give a and b such that the quantum measurement outcome from operator $M_0 = a|0\rangle\langle 0| + b|1\rangle\langle 1|$ acting on $|\psi_{AB}\rangle$ produces the post-measurement result $(|00\rangle + |11\rangle)/\sqrt{2}$ with probability 1/4. (15 points)
   Answer:
   Post-measurement result(up to normalization) is

$$M_0|\psi_{AB}\rangle = (\sqrt{3}a|00\rangle + b|11\rangle)/2 \propto |00\rangle + |11\rangle)/\sqrt{2}$$

   Therefore,

$$\sqrt{3}a = b$$

   With probability

$$\langle\psi_{AB}|M_0^\dagger M_0|\psi_{AB}\rangle = (3|a|^2 + |b|^2)/4 = 1/4$$

   so,

$$3|a|^2 + |b|^2 = 1$$

Therefore

$$a = \frac{1}{\sqrt{6}} e^{i\theta}, b = \frac{1}{\sqrt{2}} e^{i\theta}$$

where $e^{i\theta}$ is an arbitrary phase.

(b) Give an operator M_1 such that $M_0^\dagger M_0 + M_1^\dagger M_1 = I$. With what probability does the corresponding outcome occur, acting on $|\psi_{AB}\rangle$, and what is the post-measurement result? (10 points)
Answer:

$$M_0^\dagger M_0 = \frac{1}{6}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

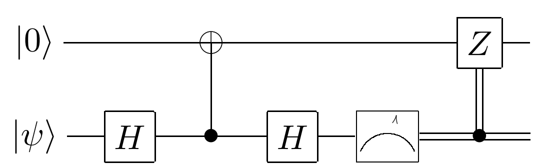
M_1 can be chosen as

$$M_1 = \sqrt{\frac{5}{6}}|0\rangle\langle 0| + \frac{1}{\sqrt{2}}|1\rangle\langle 1|$$

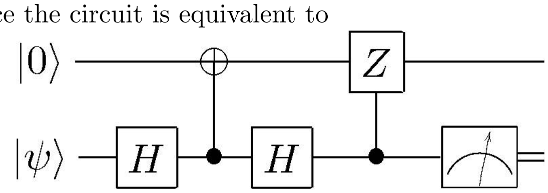
so that $M_0^\dagger M_0 + M_1^\dagger M_1 = I$
The probability for measurement result to be 1 is $1 - 1/4 = 3/4$(can also be calculated from $\langle\psi_{AB}|M_1^\dagger M_1|\psi_{AB}\rangle$).
The post measurement state is

$$M_1|\psi_{AB}\rangle / \sqrt{\langle\psi_{AB}|M_1^\dagger M_1|\psi_{AB}\rangle} = \sqrt{\frac{5}{6}}|00\rangle + \sqrt{\frac{1}{6}}|11\rangle$$

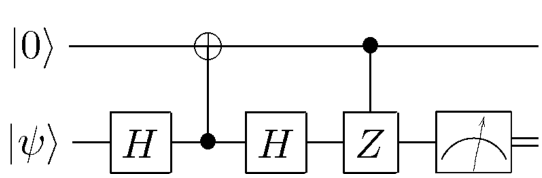
3. **Quantum Circuits**. Let $|\psi\rangle = a|0\rangle + b|1\rangle$ and consider this circuit (recall that double lines represent classical bits):



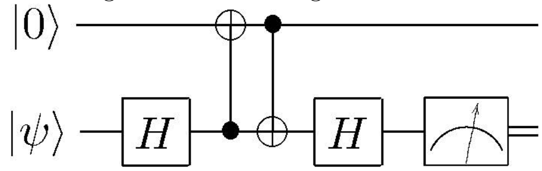
(a) What is the state $|\phi\rangle$ of the top qubit output? (15 points if you use circuit equivalences, 10 points if you work it out with states)
Answer:
Classical control operation after measurement is equivalent to quantum control operation before measurement, hence the circuit is equivalent to



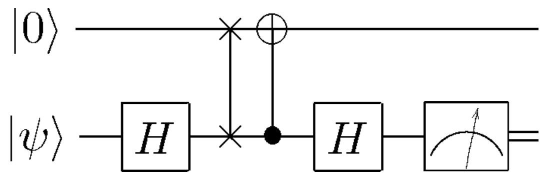
Control-Z operation is symmetric between control bit and target bit, hence the circuit is equivalent to



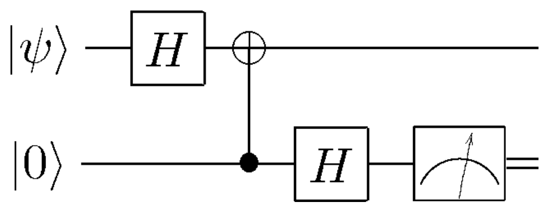
Commuting control-Z through Hadamard we get control-not



Three control-nots with alternating control and target qubit gives swap gates, therefore the circuit is equivalent to



which is equal to

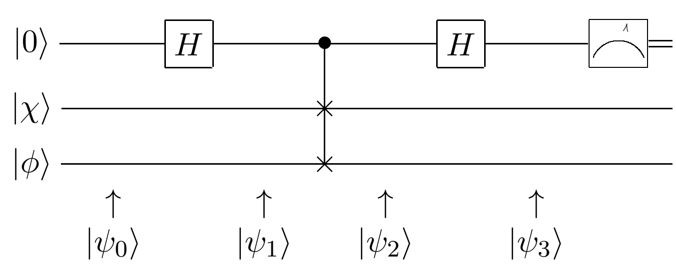


The control-not operation does not change the state as it has $|0\rangle$ as control qubit, therefore it is easy to see that the output qubit on the top line is $|\phi\rangle = H|\psi\rangle = \frac{a+b}{\sqrt{2}}|0\rangle + \frac{a-b}{\sqrt{2}}|1\rangle$.

(Hint: if working out with states, it is only necessary to find the output when $|\psi\rangle = |0\rangle$ and $|\psi\rangle = |1\rangle$. For a general input $|\psi\rangle$, the output will be a linear superposition of the output for $|\psi\rangle = |0\rangle$ and $|\psi\rangle = |1\rangle$.)

(b) What operation relates $|\phi\rangle$ to $|\psi\rangle$? (10 points)
Answer:
The Hadamard operation, H.

4. **Qubit tests**. Consider the following three-qubit quantum circuit, in which $|\chi\rangle$ and $|\phi\rangle$ are arbitrary qubits:



(a) Give the intermediate states of the circuit, $|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$. (4 points each)
Answer:

$$|\psi_0\rangle = |0\rangle|\chi\rangle|\phi\rangle$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\chi\rangle|\phi\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}|0\rangle|\chi\rangle|\phi\rangle + \frac{1}{\sqrt{2}}|1\rangle|\phi\rangle|\chi\rangle$$

$$|\psi_3\rangle = \frac{1}{2}(|0\rangle + |1\rangle)|\chi\rangle|\phi\rangle + \frac{1}{2}(|0\rangle - |1\rangle)|\phi\rangle|\chi\rangle = \frac{1}{2}|0\rangle(|\chi\rangle|\phi\rangle + |\phi\rangle|\chi\rangle) + \frac{1}{2}|1\rangle(|\chi\rangle|\phi\rangle - |\phi\rangle|\chi\rangle)$$

(b) If the measurement result is zero (ie the top qubit is $|0\rangle$), what is the state of the bottom two qubits? (4 points)

Answer:

$$\frac{|\chi\rangle|\phi\rangle + |\phi\rangle|\chi\rangle}{\sqrt{2(1 + |\langle\chi|\phi\rangle|^2)}}$$

(c) If $|\langle\chi|\phi\rangle| = \alpha$, with what probability is the measurement result zero? (10 points)

Answer:

The measurement is going to result in 0 with probability

$$\frac{1}{4}(\langle\chi|\langle\phi| + \langle\phi|\langle\chi|)(|\chi\rangle|\phi\rangle + |\phi\rangle|\chi\rangle) = \frac{1+\alpha^2}{2}$$

(You can check that the probability of measuring in 1 is $\frac{1-\alpha^2}{2}$, so the total probability adds up to 1.)