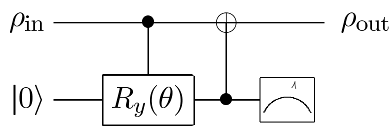
Problem Set # 10 Solutions

P1: (**Circuit models of quantum operations**) Let $\rho_{\text{out}} = \sum_k E_k \rho_{\text{in}} E_k^\dagger$.

(a) Give the operation elements E_k describing the mapping for this circuit:



What physical process does this describe?

Answer:
We use the identity $E_k = \langle k_E | U |0_E\rangle$, and as a placeholder we say $|\psi\rangle = a|0\rangle + b|1\rangle$.

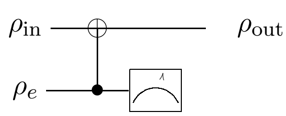
$$U|\psi\rangle|0_E\rangle = a|00\rangle + b\cdot cos(\frac{\theta}{2})|10\rangle + b\cdot sin(\frac{\theta}{2})|01\rangle$$

$$\langle 0_E| U|\psi\rangle|0_E\rangle = a|0\rangle + b\cdot cos(\frac{\theta}{2})|1\rangle\,; \qquad E_0 = \begin{pmatrix}1 & 0\\ 0 & cos(\frac{\theta}{2})\end{pmatrix}$$

$$\langle 1_E| U|\psi\rangle|0_E\rangle = b\cdot sin(\frac{\theta}{2})|0\rangle\,; \qquad E_1 = \begin{pmatrix}0 & sin(\frac{\theta}{2})\\ 0 & 0\end{pmatrix}$$

This is called an amplitude damping channel, as it reduces the amplitude of a $|1\rangle$ state.

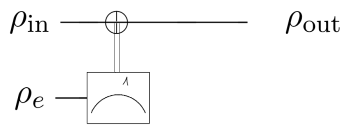
(b) Give the E_k for this circuit, assuming $\rho_e = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$:



What physical process does this describe?

Answer:

The above circuit is equivalent to this one:



Which applies X to the first qubit with probability $1-p$ and leaves it alone with probability p. Therefore,

$$E_0 = \sqrt{p} \cdot I, \qquad E_1 = \sqrt{1-p} \cdot X$$

This is called a bit-flip channel, since it flips the qubit with a certain probability.

P2: (**Quantum noise and codes**) Single qubit quantum operations $\mathcal{E}(\rho)$ model quantum noise which is corrected by quantum error correction codes.

(a) Construct operation elements for $\mathcal{E}$ such that upon input of any state ρ replaces it with the completely randomized state $I/2$. It is amazing that even such noise models as this may be corrected by codes such as the Shor code!

Answer:

We apply I, X, Y, Z gates with equal probability;

$$E_0 = \frac{I}{2}, \qquad E_1 = \frac{X}{2}, \qquad E_2 = \frac{Y}{2}, \qquad E_3 = \frac{Z}{2}.$$

We may describe the density matrix of one qubit as $\rho = I/2 + aX + bY + cZ$. Since I commutes with all operators, this component is left alone. Since X, Y, Z commute with half of the E_k and anticommute with half, these components will go to zero in the outputted density matrix, for example:

$$\mathcal{E}(X) = \sum_k E_k X E_k^\dagger = \frac{IXI + XXX + YXY + ZXZ}{4} = \frac{X + X - X - X}{4} = 0,$$

and likewise for Y, Z. Therefore, by linearity for all inputs ρ:

$$\mathcal{E}(\rho) = I/2.$$

(b) The action of the bit flip channel can be described by the quantum operation $\mathcal{E}(\rho) = (1-p)\rho + pX\rho X$. Show that this may be given an alternate operator-sum representation, as $\mathcal{E}(\rho) = (1-2p)\rho + 2pP_+\rho P_+ + 2pP_-\rho P_-$ where P_+ and P_- are projectors onto the $+1$ and -1 eigenstates of X, $(|0\rangle + |1\rangle)/\sqrt{2}$ and $(|0\rangle - |1\rangle)/\sqrt{2}$, respectively. This latter representation can be understood as a model in which the qubit is left alone with probability $1-2p$, and is 'measured' by the environment in the $|+\rangle, |-\rangle$ basis with probability $2p$.

Answer:

We use the identities $X = P_+ - P_-$, $I = P_+ + P_-$. Doing some algebra,

$$\begin{aligned} \mathcal{E}(\rho) = (1-p)\rho + pX\rho X = (1-2p)\rho + p\left(X\rho X + I\rho I\right) = \\ (1-2p)\rho + p\left(\left(P_+ - P_-\right)\rho\left(P_+ - P_-\right) + \left(P_+ + P_-\right)\rho\left(P_+ + P_-\right)\right) = \\ (1-2p)\rho + 2pP_+\rho P_+ + 2pP_-\rho P_-, \end{aligned}$$

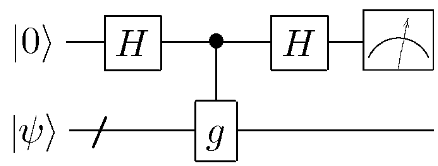
which is equivalent to measuring the state in the $|+\rangle, |-\rangle$ basis with probability $2p$.

P3: (**Shor's 9 qubit code**) The Shor code is able to protect against phase flip and bit flip errors on any qubit.

(a) Show that the syndrome measurement for detecting phase flip errors in the Shor code corresponds to measuring the observables $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$.

Answer:

The codespace of the Shor code is stabilized by $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$. Any phase flip operation Z_j anti-commutes with one or both of these operators, and can thus be detected by measuring these observables. For example, if $X_4X_5X_6X_7X_8X_9 |\psi\rangle = |\psi\rangle$, then $X_4X_5X_6X_7X_8X_9 (Z_4 |\psi\rangle) = -Z_4 |\psi\rangle$. To measure an observable g in the Pauli group , we simply apply the circuit:



We can verify that we measure $|0\rangle$ when $g|\psi\rangle = |\psi\rangle$, and $|1\rangle$ when $g|\psi\rangle = -|\psi\rangle$.

(b) Show that recovery from a phase flip on any of the first three qubits may be accomplished by applying the operator $Z_1Z_2Z_3$.

Answer:

Since the codespace of the Shor code is stabilized by the operators Z_1Z_2, Z_2Z_3, and Z_1Z_3, applying a phase flip operation of Z_1, Z_2, or Z_3 to any state in the code space is equivalent to applying the operator $Z_1Z_2Z_3$. Therefore, applying $Z_1Z_2Z_3$ will reverse that operation.

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Quantum Information Science I

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Problem Set #10

(due in class, THURSDAY 02-Dec-10)

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P4: (**Recent quantum communication results**) . Find a recent paper in the literature about a recent (post-2008) theoteical or experimental advance in quantum communication, involving distributed qubits. Write a short (< 500 word) summary of it, on the QIS wiki. See instructions on the course homepage, `http://web.mit.edu/2.111/`