MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Quantum Information Science I

December 9, 2010

Problem Set #11
(OPTIONAL – no due date)

P1: (**Entropy, Fidelity, and Amplitude Damping**) In this problem, we compute two measures of how a bad a qubit state becomes as it traverses through a common quantum noise channel, the ampltidue damping channel. Recall that the ampltiude damping channel for a single qubit is described by $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$, where the operation elements are

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \qquad E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} . \tag{1}$$

We shall let $\gamma = 1 - e^{-t/T_1}$, where t is time and T_1 is the amplitude damping time constant.

(a) Let $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Compute $F(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|))$ and plot as a function of t.

(b) Recall that $S(\rho) = -\rho \log_2 \rho$ is the von Neumann entropy of a state ρ. Plot $S(\mathcal{E}(|\psi\rangle\langle\psi|))$ as a function of t. Recall that a pure state has $S = 0$; it would seem that adding noise to a system monotonically increases its entropy. What happens in this case, and why?

(c) Find the state $|\phi(t)\rangle$ which minimizes $F(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|))$ at each point in time, and plot this minimum value as a function of time.

P2: (**Exercises with entropy**) With three games remaining in the baseball season, the Reds and the Giants are tied for first place. Depending on the outcomes of the final three games, which are played against other teams, either the Reds or the Giants will win the championship or they will tie.

(a) Assume that each of the remaining games played by the contending teams are won or lost with probability 1/2. Compute the entropy of the random variable whose outcomes are { win, lose, tie} for the Reds.

(b) Assume that the Giants win all three games against their feeble opponents. Compute the conditional entropy of the random variable whose outcomes are {win, lose, tie} for the Reds.

P3: (**Von Neumann entropy of a qubit**) Let ρ be a single qubit state (a density matrix), and recall the Bloch sphere representation

$$\rho = \frac{I + \sum_{k=1}^{3} r_k \sigma_k}{2} , \tag{2}$$

where σ_k are the three Pauli matrices. Express $S(\rho) = -\mathrm{tr}(\rho \log \rho)$ in terms of $\vec{r} = (r_x, r_y, r_z)$ and the binary entropy function $H(p) = -p \log p - (1-p)\log(1-p)$.

P4: (**Typical sequences (computational)**) Let $X_1, X_2, X_3, \ldots$ be an i.i.d sequence of random variables $\mathcal{X}$ with range $\{a, b, c\}$ and probability mass function $p(a) = 0.8$, $p(b) = p(c) = 0.1$.

(a) Calculate the entropy rate $H(\mathcal{X}) = H(X_1)$.

(b) The set of ϵ-typical sequences of length n, $A_\epsilon^{(n)}$, consists of sequences for which the number n_a of occurrences of the value a is close to the expected value $0.8n$. Find inequalities that tell when a sequence is ϵ-typical in terms of ϵ, n, and n_a.

(c) Let $A_{0.1}^{(100)}$ be the set of 0.1-typical sequences of length 100. Compute $\Pr(A_{0.1}^{(100)})$.

(d) Compute $|A_{0.1}^{(100)}|$, the number of typical sequences, and the number of bits needed to represent all typical sequences.

P5: (**Holevo's Theorem**) Suppose Alice sends Bob an equal mixture of the four pure states

$$|X_1\rangle = |0\rangle \tag{3}$$

$$|X_2\rangle = \sqrt{\frac{1}{3}}\left[|0\rangle - \sqrt{2}|1\rangle\right] \tag{4}$$

$$|X_3\rangle = \sqrt{\frac{1}{3}}\left[|0\rangle - \sqrt{2}e^{2\pi i/3}|1\rangle\right] \tag{5}$$

$$|X_4\rangle = \sqrt{\frac{1}{3}}\left[|0\rangle - \sqrt{2}e^{4\pi i/3}|1\rangle\right] . \tag{6}$$

(a) Compute the maximum mutual information between Bob's measurement and Alice's transmission.

(b) A generalized quantum measurement scheme (aka POVM) which achieves $I_0 \approx 0.415$ bits exists. That is, there are a set of operators M_k such that $\sum_k M_k = I$, and for $\rho = (1/4)\sum_k |X_k\rangle\langle X_k|$, the measurement result $y = \mathrm{tr}(\rho M_k)$ has mutual information I_0 with Alice's transmission. Give M_k.

(c) Give a (short, understandble) proof that the Holevo bound is not achievable in this sceneraio.