Problem Set # 3 Solutions

1. **Measurements and Uncertainty**

   (a) The expectation value of M on state $|\psi\rangle$ will be m, the standard deviation will be 0.

   (b) Measuring X on the state $|0\rangle$, we will get results of 1,-1 with equal probability. Therefore the expectation value is 0 and the standard deviation is 1.

2. Entropy of quantum states

   (a) The entropy of $\rho_0 = |0\rangle\langle 0|$ is $-\log_2(1) = 0$

   (b) The entropy of $\rho_1 = \frac{(|0\rangle\langle 0|+|1\rangle\langle 1|)}{2}$ is $-\frac{1}{2}\log_2(\frac{1}{2}) - \frac{1}{2}\log_2(\frac{1}{2}) = 1$

   (c) If $Tr(\rho^2) = 1$,

$$\sum_k \lambda_k^2 = \sum_k \lambda_k = 1$$

   Therefore,

$$\sum_k \lambda_k (\lambda_k - 1) = 0$$

   Since $0 \le \lambda_k \le 1$, $\forall k$, we know that $\lambda_k(\lambda_k - 1) \ge 0$, $\forall k$, and thus the only way for the above condition to be satisfied is for $\lambda_k = 0, 1$, $\forall k$. Therefore $Tr(\rho^2) = 1$ if and only if ρ has a single eigenvalue of 1 with all other eigenvalues 0.

$$S(\rho) = -\sum_k \lambda_k \log_2(\lambda_k) = 0$$

   Since $0 \le \lambda_k \le 1$, $\forall k$, we know that $\lambda_k \log_2(\lambda_k) \ge 0$, $\forall k$. Therefore, the only way for the above condition to be satisfied is for $\lambda_k = 0, 1$, $\forall k$, and thus $S(\rho) = 1$ if and only if ρ has a single eigenvalue of 1 with all other eigenvalues 0.

   Therefore, for density matrices, $Tr(\rho^2) = 1$ and $S(\rho) = 1$ are equivalent statements.

3. (a) A state is a product state if and only if it can be represented as $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$. If a state has a Schmidt number 1, it can be represented as a product state $\sum_k \sqrt{\lambda_k} |k_A\rangle |k_B\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ since only one Schmidt coefficient is nonzero. If it has a Schmidt number greater than 1, it has no such representation as $|\psi_A\rangle \otimes |\psi_B\rangle$, because if it did it would have a Schmidt number of 1 through the above representation.

(b) **Lemma:** If an entangled state between Alice and Bob has the Schmidt decomposition

$$\sum_k \sqrt{\lambda_k} |k_A\rangle |k_B\rangle$$

Then Alice's reduced density matrix is

$$\rho^A = \sum_k \lambda_k |k_A\rangle \langle k_A|$$

(Likewise for Bob)
Therefore, if $|\psi\rangle$ has a Schmidt number of 1, the reduced density matrices ρ^A, ρ^B have only one non-zero eigenvalue and are pure states. If $|\psi\rangle$ has a Schmidt number greater than 1, the reduced density matrices ρ^A, ρ^B have multiple non-zero eigenvalues and are mixed states.
Proof Of Lemma(Less Mathematical) If Bob measured his state in the $\{|k_B\rangle\}$ basis, with probability λ_k he will measure $|k_B\rangle$ and Alice's state will collapse to $|k_A\rangle$. Therefore, since the outcome of Alice's measurement can't be effected by whether Bob made his measurement, we may say:

$$\rho^A = \sum_k \lambda_k |k_A\rangle \langle k_A|$$

Proof of Lemma(More Mathematical): We may write the global density matrix as

$$\rho^{total} = \sum_{k,K'} \sqrt{\lambda_k \lambda_{k'}} |k_A\rangle_A |k_B\rangle_B \langle k'_B|_B \langle k'_A|_A$$

Alice's reduced density matrix can be obtained by taking the partial trace

$$\rho^A = Tr_B(\rho^{total}) = \sum_{k,K'} \sqrt{\lambda_k \lambda_{k'}} |k_A\rangle \langle k'_A| Tr\left(|k_B\rangle \langle k'_B|\right)$$

Since $Tr\left(|k_B\rangle \langle k'_B|\right) = \langle k'_B| |k_B\rangle = \delta_{kk'}$, we may say

$$\rho^A = \sum_k \lambda_k |k_A\rangle \langle k_A|$$

4. (a) The Schmidt decomposition of $|\phi_1\rangle = \frac{|00\rangle+|11\rangle+|22\rangle}{\sqrt{3}}$ is $\sum_{k=0,1,2} \frac{1}{\sqrt{3}} |k\rangle |k\rangle$ by inspection(Other Schmidt decompositions are also possible)

(b) The Schmidt decomposition of $|\phi_2\rangle = \frac{|00\rangle+|01\rangle+|10\rangle+|11\rangle}{2}$ is $|+\rangle |+\rangle$ by inspection, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

(c) The Schmidt decomposition of $|\phi_3\rangle = \frac{|00\rangle+|01\rangle+|10\rangle-|11\rangle}{2}$ is $\frac{1}{\sqrt{2}}(|0\rangle |+\rangle + |1\rangle |-\rangle)$ by inspection, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ (Other Schmidt decompositions are also possible)

(d) To find the schmidt decomposition of $|\phi_4\rangle = \frac{|00\rangle+|01\rangle+|11\rangle}{\sqrt{3}}$, we use the lemma proved in the previous problem. We can see that Alice's reduced density matrix is

$$\rho^A = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

having eigenvalues $\frac{1}{6}\left(3+\sqrt{5}\right) \approx .87, \frac{1}{6}\left(3-\sqrt{5}\right) \approx .13$, eigenvectors

$$\frac{\left(-1+\frac{1}{2}\left(3+\sqrt{5}\right),1\right)}{\sqrt{1+\left(-1+\frac{1}{2}\left(3+\sqrt{5}\right)\right)^2}} \approx (.85,.52), \frac{\left(-1+\frac{1}{2}\left(3-\sqrt{5}\right),1\right)}{\sqrt{1+\left(-1+\frac{1}{2}\left(3-\sqrt{5}\right)\right)^2}} \approx (-.52,.85)$$

Likewise, Bob's reduced density matrix is

$$\rho^B = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

having eigenvalues $\frac{1}{6}\left(3+\sqrt{5}\right) \approx .87, \frac{1}{6}\left(3-\sqrt{5}\right) \approx .13$, eigenvectors

$$\frac{\left(1,-1+\frac{1}{2}\left(3+\sqrt{5}\right)\right)}{\sqrt{1+\left(-1+\frac{1}{2}\left(3+\sqrt{5}\right)\right)^2}} \approx (.52,.85), \frac{\left(-1,1-\frac{1}{2}\left(3-\sqrt{5}\right)\right)}{\sqrt{1+\left(-1+\frac{1}{2}\left(3-\sqrt{5}\right)\right)^2}} \approx (-.85,.52)$$

Therefore, the Schmidt decomposition will be

$$|\phi_4\rangle = \frac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}} = \sqrt{\lambda_1}\,|\psi_{A1}\rangle\,|\psi_{B1}\rangle + \sqrt{\lambda_2}\,|\psi_{A2}\rangle\,|\psi_{B2}\rangle$$

Where $\lambda_1 = \frac{1}{6}\left(3+\sqrt{5}\right) \approx .87, \lambda_2 = \frac{1}{6}\left(3-\sqrt{5}\right) \approx .13$
And the Schmidt vectors $|\psi_{A1}\rangle, |\psi_{A2}\rangle, |\psi_{B1}\rangle, |\psi_{B2}\rangle$ are defined as above

5. (a) $\psi_1 = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$
   $\psi_2 = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\phi}|1\rangle\right)$
   $\psi_3 = \frac{1}{2}\left(\left(1+e^{i\phi}\right)|0\rangle + \left(\left(1-e^{i\phi}\right)|1\rangle\right)$

   (b) The probability of measuring a qubit to be 1 is $p = \frac{1}{4}|1-e^{i\phi}|^2 = \frac{1-\cos(\phi)}{2}$

   (c) We consider the random variable X which is 0 if we measure 0 and 1 if we measure 1. $\langle X\rangle = \langle X^2\rangle = \frac{1-cos(\phi)}{2}$ Therefore, the variance of a single measurement is

$$\langle X^2\rangle - \langle X\rangle^2 = \frac{1-cos(\phi)}{2} - \left(\frac{1-cos(\phi)}{2}\right)^2 = \frac{\sin(\phi)^2}{4}$$

   Thus the variance in the number of 1's you get after n measurements is $n\frac{\sin(\phi)^2}{4}$ and thus the standard deviation of measured probability after n experiments will be $\frac{|\sin(\phi)|}{2\sqrt{n}}$

   Since $\frac{dp}{d\phi} = \frac{\sin(\phi)}{2}$, the accuracy of the estimate for ϕ is $\Delta\phi = \frac{\Delta p}{dp/d\phi} = \frac{\frac{|\sin(\phi)|}{2\sqrt{n}}}{\left|\frac{\sin(\phi)}{2}\right|} = \frac{1}{\sqrt{n}}$

   (Note that it's impossible to tell the sign of ϕ in this way, you need to measure in a different basis to do that)