MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Quantum Information Science I

September 23, 2010

Problem Set #3
(due in class, 30-Sep-10)

1. **Measurements and uncertainty**.
   (a) Suppose we prepare a quantum system in an eigenstate $|\psi\rangle$ of some observable M, with corresponding eigenvalue m. What is the average observed value of M, and the standard deviation?
   (b) Suppose we have qubit in the state $|0\rangle$, and we measure the observable X (i.e. σ_x). What is the average value of X? What is the standard deviation of X?

2. **Entropy of quantum states**. The entropy of a quantum state, expressed as a density matrix ρ, is $S(\rho) = -\text{tr}(\rho \log_2 \rho)$; in terms of its eigenvalues λ_k, this is $S(\rho) = -\sum_k \lambda_k \log_2 \lambda_k$.
   (a) Give the entropy $S(\rho_0)$ for $\rho_0 = |0\rangle\langle 0|$.
   (b) Give the entropy of $\rho_1 = (|0\rangle\langle 0| + |1\rangle\langle 1|)/2$.
   (c) A state ρ is a pure state if and only if $\text{tr}(\rho^2) = 1$. Prove that this is equivalent to $S(\rho) = 0$. You may use the fact that ρ is a valid density matrix if and only if $\text{tr}(\rho) = 1$ and ρ is a positive operator (i.e. its eigenvalues are ≥ 0).

3. **Product states and Schmidt numbers**. Prove that a state $|\psi\rangle$ of a composite system AB is a product state if and only if it has Schmidt number 1. Prove that $|\psi\rangle$ is a product state if and only if ρ^A (and thus ρ^B) are pure states.

4. **Schmidt decompositions**. Consider a composite system consisting of two qubits. Find the Schmidt decompositions of the states

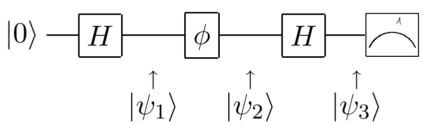
$$|\phi_1\rangle = \frac{|00\rangle + |11\rangle + |22\rangle}{\sqrt{3}} \quad (1)$$

$$|\phi_2\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} \quad (2)$$

$$|\phi_3\rangle = \frac{|00\rangle + |01\rangle + |10\rangle - |11\rangle}{2} \quad (3)$$

$$|\phi_4\rangle = \frac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}} . \quad (4)$$

5. **Interferometers**. Consider this single qubit model of an interferometer, where the goal is to estimate an unknown phase ϕ:



   Let the box with ϕ map $|0\rangle \to |0\rangle$ and $|1\rangle \to e^{i\phi}|1\rangle$.
   (a) Give the states $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$.
   (b) What is the probability p of measuring the final qubit to be one?
   (c) If this is experiment is repeated n times, what is the standard deviation Δp of the value estimated for p from the measurement results? Also give the uncertainty in the resulting estimate for ϕ, $\Delta\phi = \Delta p / |dp/d\phi|$.