MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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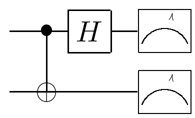
Quantum Information Science I

September 30, 2010

Problem Set #4 Solution
(due in class, 07-Oct-10)

1. **Measurement in the Bell basis** Show that the circuit



performs a measurement in the basis of the Bell states. Specifically, show that this circuit results in a measurement being performed with four operators $\{M_k\}$ such that $M_k^\dagger M_k$ are the four projectors onto the Bell states.

Answer:

The action of the circuit on any input state $|\phi\rangle$ is

$$|\phi\rangle \to CNOT|\phi\rangle \to (H_1 \otimes I_2)CNOT|\phi\rangle \to M_k(H_1 \otimes I_2)CNOT|\phi\rangle \tag{1}$$

where $M_1 = |00\rangle\langle 00|$,$M_2 = |01\rangle\langle 01|$, $M_3 = |10\rangle\langle 10|$, $M_4 = |11\rangle\langle 11|$.(Normalization factor is ignored.)

Combining $(H_1 \otimes I_2)CNOT$ into M_k, we see that the total action is equivalent to the following four operators

$$M_1' = |00\rangle(\langle 00|(H_1 \otimes I_2)CNOT) = \frac{1}{\sqrt{2}}|00\rangle(\langle 00| + \langle 11|) \tag{2}$$

$$M_2' = |01\rangle(\langle 01|(H_1 \otimes I_2)CNOT) = \frac{1}{\sqrt{2}}|01\rangle(\langle 01| + \langle 10|) \tag{3}$$

$$M_3' = |10\rangle(\langle 10|(H_1 \otimes I_2)CNOT) = \frac{1}{\sqrt{2}}|10\rangle(\langle 00| - \langle 11|) \tag{4}$$

$$M_4' = |11\rangle(\langle 11|(H_1 \otimes I_2)CNOT) = \frac{1}{\sqrt{2}}|11\rangle(\langle 01| - \langle 10|) \tag{5}$$

Therefore, it is easy to see that $(M_k')^\dagger M_k'$ are projections onto the Bell states.

$$(M_1')^\dagger M_1' = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \tag{6}$$

$$(M_2')^\dagger M_2' = \frac{1}{2}(|01\rangle + |10\rangle)(\langle 01| + \langle 10|) \tag{7}$$

$$(M_3')^\dagger M_3' = \frac{1}{2}(|00\rangle - |11\rangle)(\langle 00| - \langle 11|) \tag{8}$$

$$(M_4')^\dagger M_4' = \frac{1}{2}(|01\rangle - |10\rangle)(\langle 01| - \langle 10|) \tag{9}$$

2. **Schmidt numbers and LOCC**. Recall that the Schmidt number of a bi-partite pure state is the number of non-zero Schmidt components. Prove that the Schmidt number of a pure quantum state

cannot be increased by local unitary transforms and classical communication. The Schmidt number is strictly nonincreasing under more general conditions, namely, for *arbitrary* local operations and classical communication (LOCC); you are welcome to prove this also, but that is not required for credit. The Schmidt number is one measure of how entangled a bi-partite quantum state is.

Answer:

Suppose that a bi-partite state $|\phi\rangle_{AB}$ has Schmidt decomposition as

$$|\phi^{AB}\rangle = \sum_{i}^{n} \lambda_i |\psi_i^A\rangle|\psi_i^B\rangle \tag{10}$$

, where n is the Schmidt number, $\lambda_i > 0$ for $i = 1...n$, and $|\psi_i^A\rangle(|\psi_i^B\rangle)$ form an orthonormal set.

After local unitary transformation and classical communication, the state is changed to

$$|\tilde{\phi}^{AB}\rangle = \sum_{i}^{n} \lambda_i U^A|\psi_i^A\rangle U^B|\psi_i^B\rangle = \sum_{i}^{n} \lambda_i |\tilde{\psi}_i^A\rangle|\tilde{\psi}_i^B\rangle \tag{11}$$

Because U^A and U^B are local unitary operations, $|\tilde{\psi}_i^A\rangle$ and $|\tilde{\psi}_i^B\rangle$ still form two set of orthonormal vectors. Therefore, the above equation gives the Schmidt decomposed form of $|\tilde{\phi}^{AB}\rangle$ and the Schmidt number remains n.

If arbitrary local operations and classical communication are allowed, the state is changed into

$$|\bar{\phi}^{AB}\rangle = \sum_{i}^{n} \lambda_i M^A|\psi_i^A\rangle M^B|\psi_i^B\rangle = \sum_{i}^{n} \lambda_i |\bar{\psi}_i^A\rangle|\bar{\psi}_i^B\rangle \tag{12}$$

Here M^A, M^B are general local operations, not necessarily unitary. Therefore, $|\bar{\psi}_i^A\rangle$ and $|\bar{\psi}_i^B\rangle$ are no longer orthonormal set.

However, the reduced denstiy matrix of A still lives in the space spanned by $|\bar{\psi}_i^A\rangle$

$$\bar{\rho}^A = \text{tr}_B|\bar{\phi}^{AB}\rangle\langle\bar{\phi}^{AB}| \tag{13}$$

$$= \sum_{i}^{n}\sum_{j}^{n}\sum_{k}(\lambda_i\lambda_j\langle k|\bar{\psi}_i^B\rangle\langle\bar{\psi}_j^B|k\rangle)|\bar{\psi}_i^A\rangle\langle\bar{\psi}_j^A| \tag{14}$$

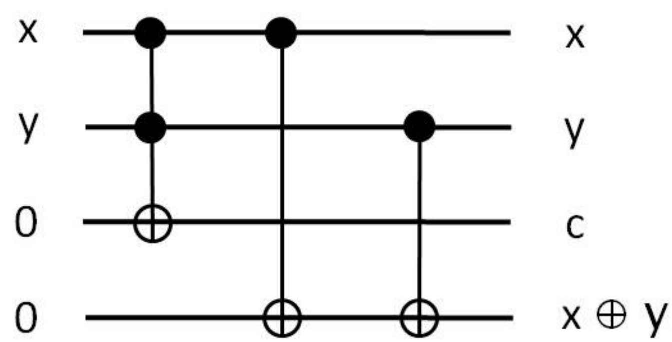
The n vectors $\{|\bar{\psi}_i^A\rangle\}$ span a space of dimension $m \leq n$. The rank n' of the reduced density matrix $\bar{\rho}^A$ is less than the dimension of the space m. Therefore $n' \leq n$.

The rank of the reduced density matrix $\bar{\rho}^A$ is equal to the Schimdt number of $|\bar{\phi}^{AB}\rangle$. Therefore, the Schimdt number of a state after arbitrary LOCC cannot be increased.

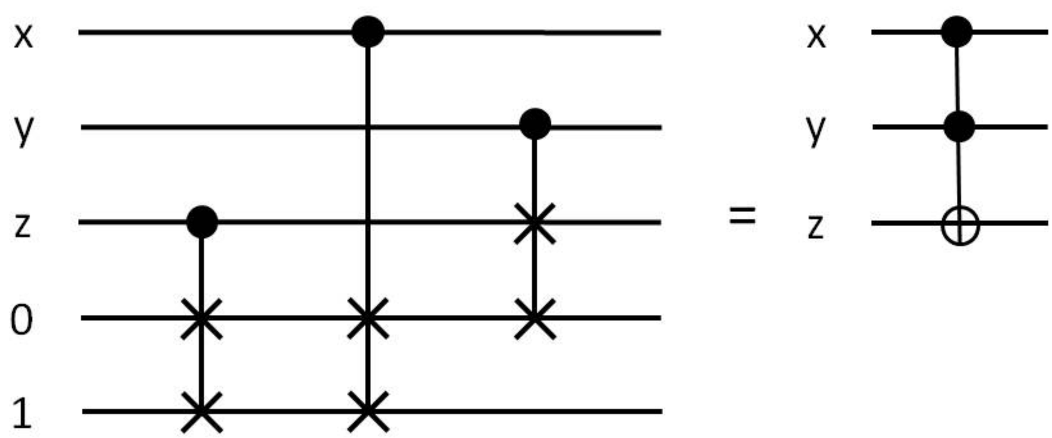
3. **Reversible circuits**.

   (a) Construct a reversible circuit which, when two bits x and y are input, outputs $(x, y, c, x \oplus y)$, where c is the carry bit when x and y are added.

   Answer:

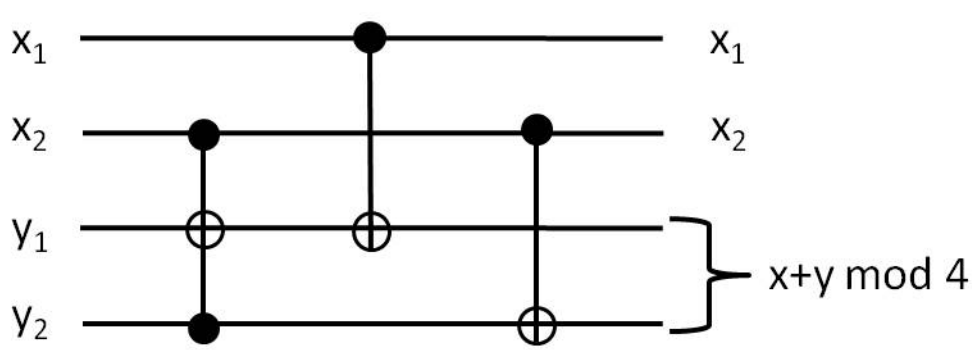


(b) Construct a reversible circuit using Fredkin gates to simulate a Toffoli gate.
Answer:



(c) Construct a quantum circuit to add two two-bit numbers x and y modulo 4. That is, the circuit should perform the transformation $|x, y\rangle \rightarrow |x, x + y \bmod 4\rangle$.
Answer:

$$x = x_1 x_2 \quad y = y_1 y_2 \tag{15}$$

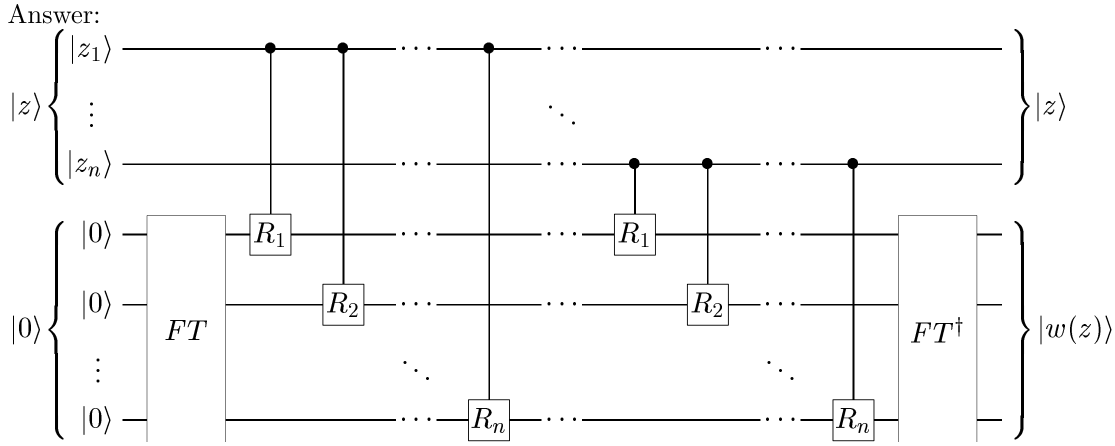


4. **Quantum circuit for the Hamming weight**. Construct a quantum circuit that performs the following unitary transformation:

$$|z\rangle|0\rangle \rightarrow |z\rangle|w(z)\rangle \,,$$

where $w(z)$ denotes the Hamming weight of z (the number of ones in its binary representation). Try to do this for the general case of z being represented by n qubits, for arbitrary n, but if you cannot think of a clever solution (which takes avantage of quantum gates, versus just classical ones), just give a circuit for $n = 3$.

Answer:



where

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{bmatrix} \tag{16}$$

'FT' is the quantum fourier transform circuit(see page 219 of book 'Quantum computation and quantum information').

A classical way to do this with $n = 3$ would be