MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Quantum Information Science I

October 28, 2010

Problem Set #7 Solution
(due in class, 04-Nov-10)

P1: (**Continuous time Grover with noise**) Consider a continuous time Grover algorithm on n qubits employing two Hamiltonians, an oracle Hamiltonian, $H_O = E_O|x\rangle\langle x|$, and a driving Hamiltonian $H_D = E_D\,|\psi\rangle\,\langle\psi|$, where $|x\rangle$ is the target state and $|\psi\rangle$ is state with all the qubits being $|0\rangle+|1\rangle$. Here, we explore the impact of a simple noise model on this algorithm.

(a) Ideally, H_D has the same strength as the oracle H_O. However, we the strength of H_D may fluctuate. Assuming $E_D = E_O(1+\delta)$, calculate the probability of being in the target state at time $\pi\sqrt{2^n}/(2E_O)$.
Answer:

$$H = H_O + H_D = E_O|x\rangle\langle x| + E_D|\psi\rangle\langle\psi|$$

$$|\psi\rangle = \left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right)^{\otimes n} = \frac{1}{\sqrt{2^n}}\sum_y |y\rangle$$

where y are binary numbers from 0 to 2^n-1.
$|\psi\rangle$ can be expanded as

$$|\psi\rangle = \frac{1}{\sqrt{2^n}}|x\rangle + \frac{\sqrt{2^n-1}}{\sqrt{2^n}}|x'\rangle$$

$|x'\rangle$ is orthogonal to $|x\rangle$.
Therefore, in the basis $|x\rangle$ and $|x'\rangle$, H can be written as

$$H = E_O\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + E_D\begin{bmatrix}\frac{1}{2^n} & \frac{\sqrt{2^n-1}}{2^n}\\ \frac{\sqrt{2^n-1}}{2^n} & \frac{2^n-1}{2^n}\end{bmatrix} = E_O\begin{bmatrix}1+(1+\delta)\frac{1}{2^n} & (1+\delta)\frac{\sqrt{2^n-1}}{2^n}\\ (1+\delta)\frac{\sqrt{2^n-1}}{2^n} & (1+\delta)\frac{2^n-1}{2^n}\end{bmatrix}$$

Writing in terms of Pauli operators,

$$H = E_O\left(\frac{2+\delta}{2}I + \left(\frac{1+\delta}{2^n} - \frac{\delta}{2}\right)Z + (1+\delta)\frac{\sqrt{2^n-1}}{2^n}X\right) = E_O(a_0 I + a_1 Z + a_2 X)$$

After time $t = \pi\sqrt{2^n}/(2E_O)$, the evolution operator is

$$exp(-iHt) = exp(-ia_0E_OtI - ia_1E_OtZ - ia_2E_OtX) = exp(-ia_0E_Ot)exp(-it'(a_1'E_OZ + a_2'E_OX))$$

where $a_1' = a_1/\sqrt{a_1^2+a_2^2}$, $a_2' = a_2/\sqrt{a_1^2+a_2^2}$, $t' = \sqrt{a_1^2+a_2^2}t$.
Hence,

$$exp(-iHt) = exp(-ia_0E_Ot)(\cos(E_Ot')I - i\sin(E_Ot')(a_1'Z + a_2'X))$$

Therefore, the state after evolution is

$$|\psi(t)\rangle = exp(-iHt)|\psi\rangle$$
$$= exp(-ia_0E_Ot)\left(\cos(E_Ot')|\psi\rangle - i\sin(E_Ot')\left(\frac{a_1' + a_2'\sqrt{2^n-1}}{\sqrt{2^n}}|x\rangle + \frac{a_2' - a_1'\sqrt{2^n-1}}{\sqrt{2^n}}|x'\rangle\right)\right)$$

The probability of being in the target state is

$$\begin{aligned} p &= |\langle x|\psi(t)\rangle|^2 \\ &= |\cos(E_O t')\langle x|\psi\rangle - i\sin(E_O t')\frac{a_1' + a_2'\sqrt{2^n-1}}{\sqrt{2^n}}|^2 \\ &= \cos^2(E_O t')/2^n + \sin^2(E_O t')\frac{(a_1' + a_2'\sqrt{2^n-1})^2}{2^n} \end{aligned}$$

To first order in δ, $\sqrt{a_1^2 + a_2^2} = \frac{1+\delta/2}{\sqrt{2^n}}$. Therefore $E_O t' = (1+\delta/2)\pi/2$.

$$\cos(E_O t') = -\pi\delta/4\ , \quad \sin(E_O t') = 1 - \pi^2\delta^2/32\ , \quad \left(\frac{a_1' + a_2'\sqrt{2^n-1}}{\sqrt{2^n}}\right)^2 = 1 - \frac{2^n-1}{4}\delta^2$$

$$p(\delta) = 1 - \delta^2(2^n-1)(\frac{1}{4} + \frac{\pi^2}{2^n 16}) = 1 - \lambda\delta^2$$

where $\lambda = (2^n-1)(\frac{1}{4} + \frac{\pi^2}{2^n 16})$.

(b) Suppose we run the algorithm on an imperfect quantum computer in which each qubit experiences a δ that has a random value given by the probability distribution $\frac{1}{\sqrt{2\pi}\sigma}\exp(-\delta^2/(2\sigma^2))$.

Calculate how many times one needs to run the algorithm to know what the target state is with probability 2/3, as a function of σ.

Answer:

The probability of getting the right answer each time is

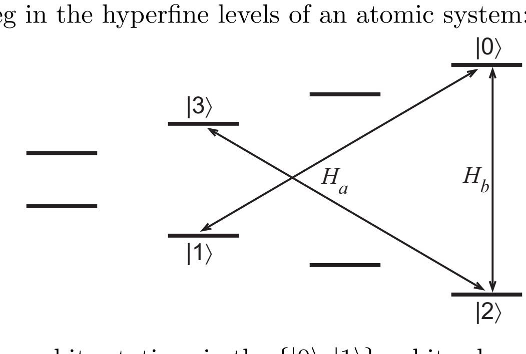
$$\int_{-\infty}^{\infty} p(\delta)\frac{1}{\sqrt{2\pi}\sigma}\exp(-\delta^2/(2\sigma^2))\,dx = 1 - \lambda\sigma^2$$

Therefore, after running the algorithm N times, the probability of NOT getting the target state goes down exponentially as $(\lambda\sigma^2)^N$. In order to succeed with probability 2/3, $N \geq log_{(\lambda\sigma^2)}\frac{1}{3}$.

P2: (**Compositions of Hamiltonian operations**) Consider a physical system with four energy levels which are addressable, $|0\rangle$, $|1\rangle$, $|2\rangle$, and $|3\rangle$. You are provided with controls which turn one of two Hamiltonians H_b, which couples $\{|0\rangle, |2\rangle\}$, and H_a, which couples $\{|2\rangle, |3\rangle\}$ and $\{|0\rangle, |1\rangle\}$. Specifically,

$$H_a(\alpha) = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ \alpha^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & \alpha^* & 0 \end{bmatrix} \qquad H_b(\beta) = \begin{bmatrix} 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 \\ \beta^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\ . \tag{1}$$

Note that α and β are both complex numbers. These may be visualized as transitions between some subset of energy levels, eg in the hyperfine levels of an atomic system:



Your mission is to perform qubit rotations in the $\{|0\rangle, |1\rangle\}$ qubit subspace, while leaving $\{|2\rangle, |3\rangle\}$ alone.

(a) Suppose H_1 and H_2 are Hamiltonians such that $\text{tr}|H_1| \leq \epsilon$ and $\text{tr}|H_2| \leq \epsilon$. Prove that $e^{-iH_1}e^{-iH_2}e^{iH_1}e^{iH_2} = e^{-iH_c} + O(\epsilon^3)$, where $H_c = i[H_1, H_2] = i(H_1H_2 - H_2H_1)$.

Answer:

Up to second order in ϵ

$$e^{-iH_1} = I - iH_1 - H_1^2/2 + O(\epsilon^3)\ ,\ \ e^{-iH_2} = I - iH_2 - H_2^2/2 + O(\epsilon^3)$$

$$e^{iH_1} = I + iH_1 - H_1^2/2 + O(\epsilon^3)\ ,\ \ e^{iH_2} = I + iH_2 - H_2^2/2 + O(\epsilon^3)$$

Retaining only up to second order in ϵ

$$\begin{aligned} e^{-iH_1}e^{-iH_2}e^{iH_1}e^{iH_2} &= (I - iH_1 - H_1^2/2)(I - iH_2 - H_2^2/2)(I + iH_1 - H_1^2/2)(I + iH_2 - H_2^2/2) \\ &= I - H_1H_2 + H_2H_1 + O(\epsilon^3) \\ &= e^{-iH_c} + O(\epsilon^3) \end{aligned}$$

(b) Show our goal is possible in principle, by constructing a set of Hamiltonians from which $H_{01} = \gamma|0\rangle\langle 1| + \gamma^*|1\rangle\langle 0|$ can be generated. Specifically, compute $H_c = i[H_b|_{\beta=1}, H_b|_{\beta=i}]/2$ and $H_d = i[H_a|_{\alpha=i}, H_c]$ and explain how to obtain H_{01} from this, by quantum simulation techniques.

Answer:

$$H_c = i[H_b|_{\beta=1}, H_b|_{\beta=i}]/2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H_d = i[H_a|_{\alpha=i}, H_c] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Define $H_e = i[H_a, H_c]$

$$H_e = i[H_a, H_c] = \begin{bmatrix} 0 & -i\alpha & 0 & 0 \\ i\alpha^* & 0 & 0 & 0 \\ 0 & 0 & 0 & i\alpha \\ 0 & 0 & -i\alpha^* & 0 \end{bmatrix}$$

H_{01} can be obtained by

$$H_{01} = (H_a|_{\alpha=\gamma} + H_e|_{\alpha=i\gamma})/2$$

Note that here we can take the sum of $H_a|_{\alpha=\gamma}$ and $H_e|_{\alpha=i\gamma}$ because they commute. $e^{itH_{01}} = e^{itH_a|_{\alpha=\gamma}/2}e^{itH_e|_{\alpha=i\gamma}/2}$.

(c) Let $R_x(\theta) = \exp\left[-i\frac{\theta}{2}(|0\rangle\langle 1| + |1\rangle\langle 0|)\right]$ be a rotation about the $\hat{x}$ axis of the $\{|0\rangle, |1\rangle\}$ qubit subspace (it acts as identity on the $\{|2\rangle, |3\rangle\}$ subspace). Give a sequence of individual Hamiltonian evolutions, eg $U = e^{it_1H_a|_{\alpha=1}}e^{it_2H_b|_{\beta=i}}\cdots$, turning on H_a and H_b sequentially (only one Hamiltonian on at a time), with specified values of α, β, and pulse durations, such that $U = R_x(\theta)$ *exactly.*

Answer:

$$H_b|_{\beta=1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = X_{02}\ ,\ \ H_b|_{\beta=-i} = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = Y_{02}$$

Composing X_{02} and Y_{02} we can get any operation on $|0\rangle$ and $|2\rangle$. For example

$$e^{iY_{02}\pi/4}e^{iX_{02}\theta/2}e^{-iY_{02}\pi/4} = e^{iZ_{02}\theta/2} = \begin{bmatrix} e^{i\theta/2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-i\theta/2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = e^{i\theta/4}U_z(-\theta/2)_{01} \oplus e^{-i\theta/4}U_z(\theta/2)_{23}$$

$$H_a|_{\alpha=-i} = Y_{01} \oplus Y_{23}\ , \quad e^{-iH_a|_{\alpha=-i}\pi/4} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}_{01} \oplus \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}_{23}$$

$$e^{iH_a|_{\alpha=-i}\pi/4}e^{iZ_{02}\theta/2}e^{-iH_a|_{\alpha=-i}\pi/4} = e^{i\theta/4}U_x(\theta/2)_{01} \oplus e^{-i\theta/4}U_x(-\theta/2)_{23} = U_1$$

But

$$e^{-iH_a|_{\alpha=1}\theta/4} = U_x(\theta/2)_{01} \oplus U_x(\theta/2)_{23} = U_2$$

Combining U_1 and U_2 we get

$$U_1U_2 = e^{i\theta/4}U_x(\theta)_{01} \oplus e^{-i\theta/4}I_{23}$$

Therefore, the total pulse sequence for applying an X operation on $|0\rangle$ and $|1\rangle$ is

$$R_x(\theta) = e^{iH_a|_{\alpha=-i}\pi/4}e^{iH_b|_{\beta=-i}\pi/4}e^{iH_b|_{\beta=1}\theta/2}e^{-iH_b|_{\beta=-i}\pi/4}e^{-iH_a|_{\alpha=-i}\pi/4}e^{-iH_a|_{\alpha=1}\theta/4}$$

(d) Do the same for $R_z(\theta) = \exp\left[-i\frac{\theta}{2}(|0\rangle\langle 0| - |1\rangle\langle 1|)\right]$, such that you now have "pulse sequences" for performing arbitrary operations on the $\{|0\rangle, |1\rangle\}$ qubit.
Answer:
Rotation around Z axis can be obtained from rotaion around X axis as

$$\begin{aligned} R_z(\theta) &= e^{iH_a|_{\alpha=-i}\pi/4}R_x(\theta)e^{iH_a|_{\alpha=-i}\pi/4} \\ &= e^{i\theta/4}(e^{iY_{01}\pi/4}U_x(\theta)_{01}e^{-iY_{01}\pi/4}) \oplus e^{-i\theta/4}(e^{iY_{23}\pi/4}I_{23}e^{-iY_{23}\pi/4}) \\ &= e^{i\theta/4}U_z(\theta)_{01} \oplus e^{-i\theta/4}I_{23} \end{aligned}$$