MIT 2.111/8.411/6.898/18.435 Quantum Information Science I *Fall, 2010*
Sam Ocko November 15, 2010

Problem Set # 8 Solutions

1. (a) The eigenvectors of $\vec{S}_1 \cdot \vec{S}_2$ are $|11\rangle, |10\rangle, \frac{|10\rangle+|01\rangle}{2}, \frac{|10\rangle-|01\rangle}{2}$ having eigenvalues $1, 1, 1, -3$. We note that these are the same as the eigenvectors of the **SWAP** operator, where they have eigenvalues $1, 1, 1, -1$. Therefore,

$$\vec{S}_1 \cdot \vec{S}_2 = 2 \cdot \mathbf{SWAP} - 1.$$

Setting $J(t) = 1$ for a time of π we apply the unitary:

$$e^{-i\frac{\pi \vec{S}_1 \cdot \vec{S}_2}{4}} = e^{-i(2\cdot\mathbf{SWAP}-1)\frac{\pi}{4}} \propto e^{-i\frac{\pi\mathbf{SWAP}}{2}} \propto \mathbf{SWAP}.$$

(b) We can implement the $\sqrt{\mathbf{SWAP}}$gate by applying the hamiltonian for half the time. To within a phase

$$\sqrt{\mathbf{SWAP}} \propto e^{-\frac{i\pi\mathbf{SWAP}}{4}} = \frac{1 - i\mathbf{SWAP}}{\sqrt{2}}.$$

To apply a ControlledT gate, we implement the following circuit:

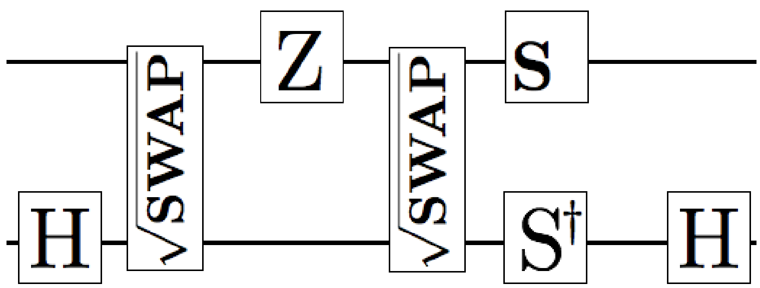

FIG. 1. Quantum circuit which gives Controlled-Not gate

To see that this circuit works, we see that

$$\sqrt{\mathbf{SWAP}}Z_1\sqrt{\mathbf{SWAP}} \propto \frac{1}{2}(1 - i\mathbf{SWAP})\, Z_1 (1 - i\mathbf{SWAP}) \propto$$

$$\frac{1}{2}(Z_1 - Z_2) - i(Z_1 + Z_2)\,\mathbf{SWAP} = \frac{1}{2}((Z_1 - Z_2) - i(Z_1 + Z_2)) =$$

$$\begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

Therefore,

$$S_1 S_2^\dagger \sqrt{\mathbf{SWAP}} Z_1 \sqrt{\mathbf{SWAP}} \propto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \cdot \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} =$$

$$\begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \propto \textbf{Cont-Z}$$

And by putting Hadamards on both sides, we turn a Controlled-Z gate into a Controlled-Not gate.

2. When $a = b = \frac{-1}{4}, c = \frac{1}{4}$, $H = \frac{(1-Z_1)(1-Z_2)}{4} - \frac{1}{4} = |11\rangle\langle 11| - \frac{1}{4}$. Turning on the Hamiltonian for a time of π we apply the unitary:

$$e^{-i\pi H} \propto e^{-i\pi|11\rangle\langle 11|} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-i\pi} \end{pmatrix} = \textbf{Cont-Z}$$

By putting Hadamards on both sides of a Controlled-Z gate, we get a Controlled-Not gate.

3. (a)

$$[H, a] = \hbar\omega\left(a^\dagger a \cdot a - a a^\dagger a\right) = \hbar\omega\left(\left(a a^\dagger - 1\right) \cdot a - a a^\dagger a\right) = -\hbar\omega a$$

Therefore, when $H\left|\psi\right\rangle = E\left|\psi\right\rangle$, we use the commutation relations $[a, a^\dagger] = 1$

$$H\left(a\left|\psi\right\rangle\right) = \left(aH - \hbar\omega a\right)\left|\psi\right\rangle = \left(aE - \hbar\omega a\right)\left|\psi\right\rangle = \left(E - \hbar\omega\right)\left(a\left|\psi\right\rangle\right)$$

so $a\left|\psi\right\rangle$ is an eigenstate with energy $E - \hbar\omega$.

(b) We can similarly show $a^\dagger\left|\psi\right\rangle$ is an eigenstate with energy $E + \hbar\omega$ using $[H, a^\dagger] = \hbar\omega a^\dagger$:

$$H\left(a^\dagger\left|\psi\right\rangle\right) = \left(a^\dagger H + \hbar\omega a^\dagger\right)\left|\psi\right\rangle = \left(a^\dagger E + \hbar\omega a^\dagger\right)\left|\psi\right\rangle = \left(E + \hbar\omega\right)\left(a^\dagger\left|\psi\right\rangle\right)$$

Therefore, using induction, $\left|n\right\rangle = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}}$ is an eigenstate with energy $n\hbar\omega$. We must now prove the normalization $\left\langle n\,|n\right\rangle = 1$. To do so, we use the identity $a\left(a^\dagger\right)^k = k\left(a^\dagger\right)^{k-1} + \left(a^\dagger\right)^k a$, and by induction:

$$\left|\left(a^\dagger\right)^n\left|0\right\rangle\right|^2 = \left\langle 0\right| a^n \left(a^\dagger\right)^n \left|0\right\rangle =$$
$$\left\langle 0\right| a^{n-1}\left(n\left(a^\dagger\right)^{n-1} + \left(a^\dagger\right)^n a\right)\left|0\right\rangle = n\left\langle 0\right| a^{n-1}\left(a^\dagger\right)^{n-1}\left|0\right\rangle = \ldots = n!\left\langle 0\,|0\right\rangle$$

Therefore, $\left\langle n\,|n\right\rangle = \frac{n!}{n!}$ =1 and is a normalized eigenstate with energy $n\hbar\omega$.

4. (a) We use the identity $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$:

$$a\,|\alpha\rangle = e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} a\,|n\rangle = e^{\frac{-|\alpha|^2}{2}} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{(n-1)!}}\,|n-1\rangle = e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^{n+1}}{\sqrt{n!}}\,|n\rangle = \alpha\,|\alpha\rangle$$

(b)

$$\langle\alpha\,|\alpha\rangle =$$
$$e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{(\alpha^*)^n}{\sqrt{n!}}\,\langle n| \sum_{n'=0}^{\infty} \frac{\alpha^{n'}}{\sqrt{n'!}}\,|n'\rangle =$$
$$e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\left(|\alpha|^2\right)^n}{n!} = 1$$

(c) We note that

$$|\alpha\rangle = e^{-\frac{|\alpha^2|}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle = e^{-\frac{|\alpha^2|}{2}} \sum_{n=0}^{\infty} \frac{\left(\alpha a^\dagger\right)^n}{n!}\,|0\rangle = e^{-\frac{|\alpha^2|}{2}} e^{a^\dagger\alpha}\,|0\rangle\,,$$

likewise for $|\beta\rangle$. We also note that

$$Ba^\dagger B^\dagger = \left(BaB^\dagger\right)^\dagger = (acos(\theta) + bsin(\theta))^\dagger = a^\dagger cos(\theta) + b^\dagger sin(\theta),$$

likewise for $Bb^\dagger B^\dagger$. Therefore, using the fact that a, b commute:

$$B\,|\alpha\rangle\,|\beta\rangle = Be^{-\frac{|\alpha^2|}{2}} e^{\alpha a^\dagger} e^{-\frac{|\beta^2|}{2}} e^{\beta b^\dagger}\,|00\rangle =$$
$$e^{-\frac{|\alpha|^2+|\beta|^2}{2}} e^{\alpha\left(a^\dagger cos(\theta)+b^\dagger sin(\theta)\right)} e^{\beta\left(-a^\dagger sin(\theta)+b^\dagger cos(\theta)\right)} B\,|00\rangle =$$
$$e^{-\frac{|\alpha cos(\theta)-\beta sin(\theta)|^2+|\alpha sin(\theta)+\beta cos(\theta)|^2}{2}} e^{a^\dagger(\alpha cos(\theta)-\beta sin(\theta))} e^{b^\dagger(\alpha sin(\theta)+\beta cos(\theta))}\,|00\rangle =$$
$$|\alpha cos(\theta) - \beta sin(\theta)\rangle \otimes |\alpha sin(\theta) + \beta cos(\theta)\rangle$$

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

MIT 2.111/8.411/6.898/18.435
Quantum Information Science I

November 4, 2010

Problem Set #8
(due in class, TUESDAY 16-Nov-10)

P1: (**Universality of Heisenberg Hamiltonian**) Consider the two-qubit Heisenberg Hamiltonian

$$H(t) = J(t)\vec{S}_1 \cdot \vec{S}_2 = \frac{J(t)}{4}\Big[X_1X_2 + Y_1Y_2 + Z_1Z_2\Big]\,. \tag{1}$$

(a) Show that a swap operation U can be implemented by turning on $J(t)$ for an appropriate amount of time τ, to obtain $U = \exp(-i\pi\vec{S}_1 \cdot \vec{S}_2)$.

(b) Compute the $\sqrt{\text{SWAP}}$ gate, obtained by turning on $J(t)$ for time $\tau/2$. Together with arbitrary single qubit gates, the $\sqrt{\text{SWAP}}$ gate is universal for quantum computation.

P2: (**NMR controlled-NOT**) Give an explicit sequence of single qubit rotations which realize a controlled-NOT between two spins evolving under the Hamiltonian $H = aZ_1 + bZ_2 + cZ_1Z_2$. Simplify the result as much as you can, to reduce the number of single qubit rotations.

P3: (**Simple harmonic oscillators**) Consider the Hamiltonian $H = \hbar\omega a^\dagger a$, and let $|n\rangle$ be an energy eigenstate with energy $n\hbar\omega$.

(a) Compute $[H, a] = Ha - aH$ and use the result to show that if $|\psi\rangle$ is an eigenstate of H with energy $E \geq n\hbar\omega$, then $a^n|\psi\rangle$ is an eigenstate with energy $E - n\hbar\omega$.

(b) Show that $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$.

P4: (**Coherent states**) The coherent state $|\alpha\rangle$ of a simple harmonic oscillator is defined as

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\,, \tag{2}$$

where $|n\rangle$ is an n-photon energy eigenstate.

(a) Prove that a coherent state is an eigenstate of the photon annihilation operator, that is, show $a|\alpha\rangle = \lambda|\alpha\rangle$ for some constant λ.

(b) Compute $\langle\alpha|\alpha\rangle$.

(c) Let $B = \exp\left[\theta\left(a^\dagger b - ab^\dagger\right)\right]$ be the beamsplitter operator coupling two simple harmonic oscillators. Using the fact that $BaB^\dagger = a\cos\theta + b\sin\theta$ and $BbB^\dagger = -a\sin\theta + b\cos\theta$, compute $B|\alpha\rangle|\beta\rangle$ where $|\alpha\rangle$ and $|\beta\rangle$ are coherent states of the two systems. Express the result in a simple form.

P5: (**Recent implementations of quantum algorithms and protocols**) . Find a recent paper in the literature about an implementation of a quantum algorithm (preferably) or a quantum protocol, involving more than one qubit. Write a short (< 500 word) summary of it, on the QIS wiki. See instructions on the course homepage, `http://web.mit.edu/2.111/`