MASSACHUSETTS INSTITUTE OF TECHNOLOGY

MIT 2.111/8.411/6.898/18.435
Quantum Information Science I

November 4, 2010

# Problem Set #8

(due in class, TUESDAY 16-Nov-10)

**P1:** (**Universality of Heisenberg Hamiltonian**) Consider the two-qubit Heisenberg Hamiltonian

$$H(t) = J(t)\vec{S}_1 \cdot \vec{S}_2 = \frac{J(t)}{4}\Big[X_1X_2 + Y_1Y_2 + Z_1Z_2\Big] \,. \tag{1}$$

(a) Show that a swap operation $U$ can be implemented by turning on $J(t)$ for an appropriate amount of time $\tau$, to obtain $U = \exp(-i\pi\vec{S}_1 \cdot \vec{S}_2)$.

(b) Compute the $\sqrt{\text{SWAP}}$ gate, obtained by turning on $J(t)$ for time $\tau/2$. Together with arbitrary single qubit gates, the $\sqrt{\text{SWAP}}$ gate is universal for quantum computation.

**P2:** (**NMR controlled-NOT**) Give an explicit sequence of single qubit rotations which realize a controlled-NOT between two spins evolving under the Hamiltonian $H = aZ_1 + bZ_2 + cZ_1Z_2$. Simplify the result as much as you can, to reduce the number of single qubit rotations.

**P3:** (**Simple harmonic oscillators**) Consider the Hamiltonian $H = \hbar\omega a^\dagger a$, and let $|n\rangle$ be an energy eigenstate with energy $n\hbar\omega$.

(a) Compute $[H, a] = Ha - aH$ and use the result to show that if $|\psi\rangle$ is an eigenstate of $H$ with energy $E \geq n\hbar\omega$, then $a^n|\psi\rangle$ is an eigenstate with energy $E - n\hbar\omega$.

(b) Show that $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$.

**P4:** (**Coherent states**) The coherent state $|\alpha\rangle$ of a simple harmonic oscillator is defined as

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \,, \tag{2}$$

where $|n\rangle$ is an $n$-photon energy eigenstate.

(a) Prove that a coherent state is an eigenstate of the photon annihilation operator, that is, show $a|\alpha\rangle = \lambda|\alpha\rangle$ for some constant $\lambda$.

(b) Compute $\langle\alpha|\alpha\rangle$.

(c) Let $B = \exp\left[\theta\left(a^\dagger b - ab^\dagger\right)\right]$ be the beamsplitter operator coupling two simple harmonic oscillators. Using the fact that $BaB^\dagger = a\cos\theta + b\sin\theta$ and $BbB^\dagger = -a\sin\theta + b\cos\theta$, compute $B|\alpha\rangle|\beta\rangle$ where $|\alpha\rangle$ and $|\beta\rangle$ are coherent states of the two systems. Express the result in a simple form.

**P5:** (**Recent implementations of quantum algorithms and protocols**) . Find a recent paper in the literature about an implementation of a quantum algorithm (preferably) or a quantum protocol, involving more than one qubit. Write a short (< 500 word) summary of it, on the QIS wiki. See instructions on the course homepage, `http://web.mit.edu/2.111/`