MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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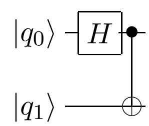
Quantum Information Science I

November 16, 2010

Problem Set #9 Solution

(due in class, TUESDAY 23-Nov-10)

P1: (**Feynman's Hamiltonian model for QC**) Read the paper *Quantum Mechanical Computers*, Optics News, February 1985, by Richard Feynman (see course website for link). Following Feynman's prescription, give a Hamiltonian for simulating this discrete quantum circuit,



using a system of interacting spins, with one or more spins acting as a clock. Describe your system and explain how your Hamiltonian simulation works.

Answer:

The system consists of five spins with $|q_0\rangle$, $|q_1\rangle$ the register qubit and $|t_0\rangle$, $|t_1\rangle$ and $|t_2\rangle$ the clock qubit. Initialize the system so that $|t_0\rangle$ is in state $|1\rangle$, $|t_1\rangle$ and $|t_2\rangle$ is in state $|0\rangle$. $|q_0\rangle$, $|q_1\rangle$ are put in appropriate initial states. Or instead of initializing $|q_0\rangle$, $|q_1\rangle$ by hand, we could include the initialization process into the Hamiltonian evolution. We could design Hamiltonian terms which take a standard initial state(all $|0\rangle$ for example) to the desired initial state for the circuit.

The simulation Hamiltonian for the two gates in the circuit is

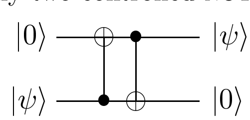
$$\sigma^+_{t_1}\sigma^-_{t_0}H_{q_0} + \sigma^+_{t_2}\sigma^-_{t_1}CNOT_{q_0,q_1} + \text{h.c.}$$

where h.c. stands for hermitian conjugate of all the previous terms. $\sigma^+ = |1\rangle\langle 0|$, $\sigma^- = |0\rangle\langle 1|$.

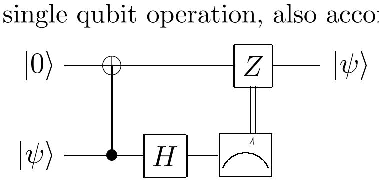
The simulation works by turning the Hamiltonian on and measuring the state of qubit $|t_2\rangle$ in the $|0\rangle$, $|1\rangle$ basis. When the measurement results in $|1\rangle$, the evolution on the register qubit will simulate the given circuit. Note that there should an adequate time interval between subsequent measurements of $|t_2\rangle$. It should not be measured all the time. Actually, if $|t_2\rangle$ is measured continuously, Zeno effect says that it will be locked in its initial state $|0\rangle$ and no evolution will take place.

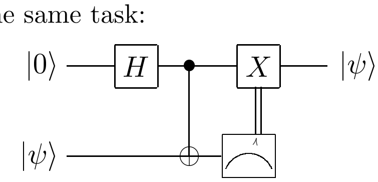
Extra clock qubits can be added before $|t_0\rangle$ or after $|t_2\rangle$ to improve the action of the computer. The extra terms added to the Hamiltonian are $\sigma^+_{t_i}\sigma^-_{t_i-1} + \text{h.c.}$, for $i \leq 0$ or $i > 2$. No evolution on the register qubits takes place with the transition between these clock qubits. With these changes, we can start the computation by putting the cursor in a wave packet before t_0. Measurement of $|t_i\rangle$, $i \geq 2$ in state $|1\rangle$ would leave the register qubit in the output state of the simulated circuit.

P2: (**Teleportation circuits**) An unknown qubit in the state $|\psi\rangle$ can be swapped with a second qubit which is prepared in the state $|0\rangle$ using only two controlled-NOT gates, with the circuit



Show that the two circuits below, which use only a single CNOT gate, with measurement and a classically controlled single qubit operation, also accomplish the same task:

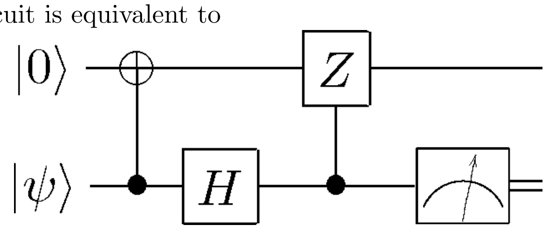




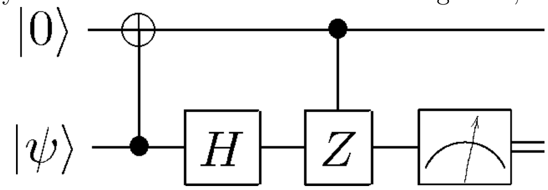
Answer:

First, consider the left hand side circuit.

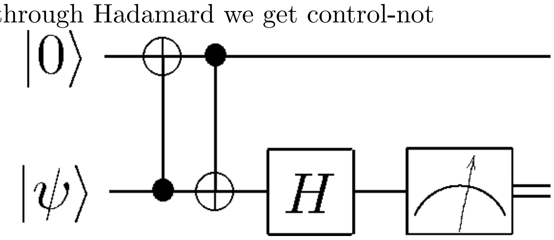
Classical control operation after measurement is equivalent to quantum control operation before measurement, hence the circuit is equivalent to



Control-Z operation is symmetric between control bit and target bit, hence the circuit is equivalent to



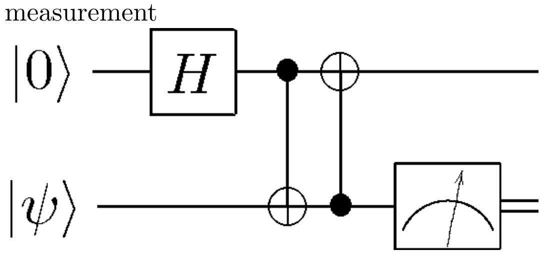
Commuting control-Z through Hadamard we get control-not



The two CNOT gates swaps the two qubits. Obviously, $|\psi\rangle$ is output on the first line.

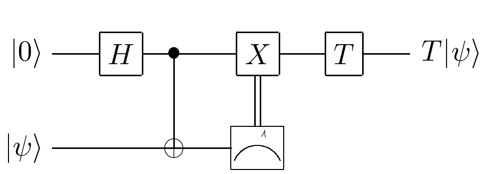
Now, let's consider the right hand side circuit.

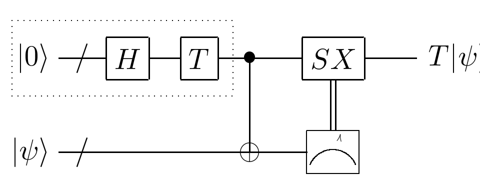
Similar to the previous case, Classical control operation after measurement can be mapped to quantum control operation before measurement



The two CNOT gates swaps the two qubits. Obviously, $|\psi\rangle$ is output on the first line.

P3: (**T gate construction using teleportation**) One way to implement a T gate is to first swap the qubit state $|\psi\rangle$ you wish to transform with some known state $|0\rangle$, then to apply a T gate to the resulting qubit (circuit A, below). Doing this does not seem particularly useful, but actually it leads to something which is! Show that by using the relations $TX = \exp(-i\pi/4)SXT$ and $TU = UT$ (U is the controlled-NOT gate, and T acts on the control qubit) we may obtain circuit B:

(A)


(B)


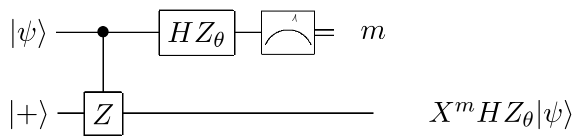
Answer:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

Commuting T back through the control-X operation. As $TX = \exp(-i\pi/4)SXT$, the controlled operation becomes SX(ignore the total phase).

Commuting T back again through U the controlled-NOT gate, U does not change. Therefore, we obtain circuit B.

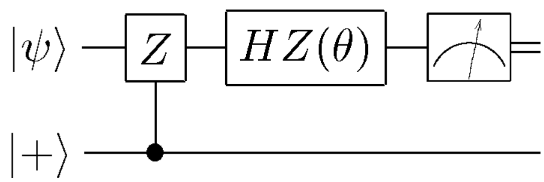
P4: (**Z-rotation gate using teleportation**) Let Z_θ denote a single qubit rotation $R_z(\theta)$ about the $\hat{z}$ axis. Prove that the output of this quantum circuit:



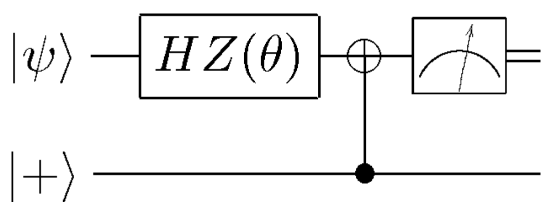
is a qubit in the state $X^m HZ_\theta|\psi\rangle$, where m is the measurement result. This is a key idea used in the cluster model of QC. How is this construction related to teleportation?

Answer:

Controlled-Z is symmetric between control and target qubit, therefore the circuit can be written as



$Z(\theta)$ commute with control-Z and contolled-Z conjugated by H is controlled-NOT. Therefore



Define $|\tilde{\psi}\rangle = HZ(\theta)|\psi\rangle$. Compare this circuit with the second one in problem 2 we can see that the output state is $|\tilde{\psi}\rangle$ up to classically controlled X operation. Therefore the output is $X^m|\tilde{\psi}\rangle = X^m HZ(\theta)|\psi\rangle$.