2.111J/18.435J/ESD.79 Quantum Computation — Due: Dec 6, in class

Problem Set 9

Problem 1

In class on Thursday, we considered open quantum system dynamics in which a system, S, interacts with the environment, E. We label the density matrix for the system and environment as ρ_{SE}. If at time 0, the density matrix is $\rho_{\mathrm{SE}}(0)$, then at time t, it would have evolved into $\rho_{\mathrm{SE}}(t) = U_{\mathrm{SE}}\rho_{\mathrm{SE}}(0)U_{\mathrm{SE}}^{\dagger}$. The density matrix of the system alone would be $\rho_{\mathrm{S}}(t) = \mathrm{tr}(\rho_{\mathrm{SE}}(t))$.

Consider now a density matrix such that the system and the environment states are totally unentangled initially,

$$\begin{aligned} \rho_{\mathrm{SE}}(0) &= \rho_{\mathrm{S}}(0) \otimes \rho_{\mathrm{E}}(0), \text{ and} \\ \rho_{\mathrm{E}}(0) &= |0\rangle_E \langle 0| , \\ |\psi(0)\rangle_{\mathrm{S}} &= \alpha |0\rangle_{\mathrm{S}} + \beta |1\rangle_{\mathrm{S}} \ . \end{aligned}$$

The nature of the interaction is $U \equiv$ CNOT, with the system as the control. Show that

$$\rho_{\mathrm{S}} = \alpha\bar{\alpha} |0\rangle \langle 0| + \beta\bar{\beta} |1\rangle \langle 1| \ .$$

In words, this means that the interaction of the system with the environment has destroyed the off-diagonal terms in ρ_{S}, which were present initially.

Problem 2

Prove that in general, we can write $\rho_{\mathrm{S}}(t) = \mathrm{tr}_{\mathrm{E}}(U\rho_{\mathrm{S}} \otimes \rho_{\mathrm{E}}U^{\dagger})$ as $\rho_{\mathrm{S}}(t) = \sum_k A_k\rho_{\mathrm{S}}(0)A_k^{\dagger}$, where $\sum_k A_k^{\dagger}A_k = I$. (Hint: To show this, write $\rho_{\mathrm{E}}(0) = \sum_i P_i |x_i\rangle \langle x_i|$ and express A_k in terms of $|x_i\rangle$ and U.)

Problem 3

Find the A_k that corresponds to complete decoherence as in the CNOT example for a qubit. (Hint: Try I, σ_x, σ_y, and σ_z and see what they do.)