2.111J/18.435J Quantum Computation Problem Set 7 Solutions
(Due: Tuesday, November 15, 2005)

1) Consider a 2 spin system possessing an Ising interaction $\hbar\Gamma(\sigma_z^A \otimes \sigma_z^B)/2$ that is placed in a magnetic field $B(t) = B\cos(\omega t)\hat{x} + B\sin(\omega t)\hat{y} + B_0\hat{z}$. Let $\omega_A = \gamma_A B_0$ and $\omega_B = \gamma_B B_0$ denote the respective Larmor frequencies of spins A and B around the z-axis. Viewed in a frame co-rotating with the circularly polarized field, the system's equation of motion is

$$\begin{aligned}\frac{d|\chi(t)\rangle}{dt} &= -\frac{i}{\hbar}H_{\mathrm{rot}}|\chi(t)\rangle \\ &= -\frac{i}{2}\left[(\omega_A-\omega)\sigma_z^A + (\omega_B-\omega)\sigma_z^B + \Gamma(\sigma_z^A\otimes\sigma_z^B) + \gamma_A B\sigma_x^A + \gamma_B B\sigma_x^B\right]|\chi(t)\rangle.\end{aligned}$$

Evaluate $e^{-itH_{\mathrm{rot}}/\hbar}$ for the case where the time-dependent part of the B field is applied at frequency $\omega = \omega_A - \Gamma$ for a time $t = \pi/(\gamma_A B)$. To what type of quantum logic operation does such time evolution correspond?

Solution:

In principle, it is possible to write down an exact, closed-form solution for $\exp(-iH_{\mathrm{rot}}t/\hbar)$. The matrix $-iH_{\mathrm{rot}}t/\hbar$, being 4×4, has a quartic characteristic equation, and the roots of any quartic equation can be given exactly in closed form. However, the resulting expression would be horrendously complicated. Thus, we need approximations, preferably ones that allow you to estimate quantitatively the resulting inaccuracies.

But before all that, here's some guiding intuition:

1. Time-dependent terms in quantum computer Hamiltonians are generally added in order to drive transitions between the eigenstates of the static part of the Hamiltonian.
2. Generally, the only significant transitions caused by a sinusoidal drive with frequency ω are those on or near resonance to ω.

Perhaps the easiest way to see what transitions are resonant or near resonant is to examine not the rotating frame, but rather the original "lab frame" where all the terms associated with the sinusoidal drive actually oscillate in time with frequency ω. Recall that moving to the frame rotating with frequency ω around the z-axis not only made the sinusoidally varying magnetic field look time-independent, but also made the Larmor frequencies associated with the the z-fields smaller by an amount ω. Thee static part of the Hamiltonian in the lab frame thus must be

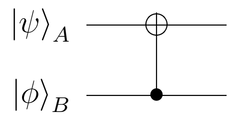
$$H_{\mathrm{static}} = \frac{\hbar}{2}\Big[\omega_A\sigma_z^A + \omega_B\sigma_z^B + \Gamma(\sigma_z^A\otimes\sigma_z^B)\Big],$$

which has eigenenergies

$$E_{00}^{(0)} = \frac{\hbar}{2}\big[\omega_A+\omega_B+\Gamma\big], \quad E_{01}^{(0)} = \frac{\hbar}{2}\big[\omega_A-\omega_B-\Gamma\big],$$

$$E_{10}^{(0)} = \frac{\hbar}{2}\big[-\omega_A+\omega_B-\Gamma\big], \quad E_{11}^{(0)} = \frac{\hbar}{2}\big[-\omega_A-\omega_B+\Gamma\big].$$

Therefore, a sinusoidal drive with frequency $\omega = \omega_A - \Gamma$ is resonant with only one transition, $|01\rangle \leftrightarrow |11\rangle$, and thus should correspond approximately to a CNOT of the form



or some undershoot or overshoot thereof.

In order to see (roughly) how well a CNOT is enacted, a nice and simple (i.e., just 2-level, not 4-level) approximation is to investigate what happens to one qubit if the other qubit were to be fixed in some particular state.

We move back to the rotating frame and consider H_{rot}. With $\omega = \omega_A - \Gamma$,

$$H_{\text{rot}} = \frac{\hbar}{2}\Big[\Gamma\sigma_z^A + (\omega_B - \omega_A + \Gamma)\sigma_z^B + \Gamma(\sigma_z^A \otimes \sigma_z^B) + \gamma_A B\sigma_x^A + \gamma_B B\sigma_x^B\Big]$$

Noting that $\sigma_z^A \otimes \sigma_z^B = \sigma_z^A \otimes |0\rangle\langle 0|_B - \sigma_z^A \otimes |1\rangle\langle 1|_B$, we see that

- If qubit B is fixed in the state $|0\rangle_B$, then the part of H_{rot} affecting qubit A is

$$H_{\text{rot}}^{A,\,\text{if}\,|0\rangle_B} = \frac{\hbar}{2}\Big[2\Gamma\sigma_z^A + \gamma_A B\sigma_x^A\Big].$$

  This causes qubit A to precess around an axis in the xz-plane of its Bloch sphere that is at an angle

$$\theta = \arcsin\left(\frac{\gamma_A B}{\sqrt{4\Gamma^2 + \gamma_A^2 B^2}}\right)$$

  clockwise from the z-axis. Thus, if qubit B is fixed in state $|0\rangle_B$ and qubit A begins along the z-axis, then qubit A will always be in a state within 2θ of the z-axis. That is, for all t, $\langle\psi_A(0)|\psi_A(t)\rangle \geq \cos(2\theta)$. More qualitatively, we conclude that if $\gamma_A B \ll 2\Gamma$, then qubit A will not appreciably change when qubit B is fixed in the state $|0\rangle_B$.

- However, if qubit B is fixed in the state $|1\rangle_B$, then the part of H_{rot} affecting qubit A is

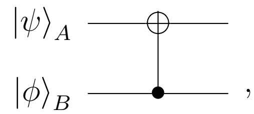
$$H_{\text{rot}}^{A,\,\text{if}\,|1\rangle_B} = \frac{\hbar\gamma_A B}{2}\sigma_x^A.$$

  This causes qubit A to precess around the x-axis, which allows full population transfer $|0\rangle_A \leftrightarrow |1\rangle_A$. The unitary matrix corresponding to this precession in the rotating frame is

$$U_{\text{rot}}^{A,\,\text{if}\,|1\rangle_B}(t) = \exp\left(\frac{-it\gamma_A B\sigma_x}{2}\right) = R_{\hat{x}}\left(\gamma_A B t\right)$$

  where $R_{\hat{x}}\left(\gamma_A B t\right)$ is a rotation of an angle $\gamma_A B t$ counterclockwise around the x-axis. Thus for $t = \pi/(\gamma_A B)$, we have $R_{\hat{x}}(\pi)$ and full population transfer $|0\rangle_A \leftrightarrow |1\rangle_A$, exactly as desired for a CNOT that acts on qubit A when qubit B is in $|1\rangle_B$.

At this point, we are only halfway done. While the above scenarios of fixing the state of qubit B and calculating what happens to qubit A are obviously key to verifying that H_{rot} applied for $t = \pi/(\gamma_A B)$ enacts unitary dynamics in the rotating frame corresponding to a CNOT of the form



it is equally important to verify that if the state of qubit A were fixed, then, regardless of A's state, there is not any significant effect on qubit B. Noting that $\sigma_z^A \otimes \sigma_z^B = |0\rangle\langle 0|_A \otimes \sigma_z^B - |1\rangle\langle 1|_A \otimes \sigma_z^B$, we see that

- If qubit A is fixed in the state $|0\rangle_A$, then the part of H_{rot} affecting qubit B is

$$H_{\text{rot}}^{B,\,\text{if}\,|0\rangle_A} = \frac{\hbar}{2}\Big[(\omega_B - \omega_A + 2\Gamma)\sigma_z^B + \gamma_B B \sigma_x^B\Big].$$

  This causes qubit B to precess around an axis in the xz-plane of its Bloch sphere that is at an angle

$$\varphi = \arcsin\left(\frac{\gamma_B B}{\sqrt{(\omega_B - \omega_A + 2\Gamma)^2 + \gamma_B^2 B^2}}\right)$$

  clockwise from the z-axis. Thus, if qubit A is fixed in state $|0\rangle_A$ and qubit B begins along the z-axis then qubit B will always be in a state within 2φ of the z-axis. That is, for all t, $\langle\psi_B(0)|\psi_B(t)\rangle \geq \cos(2\varphi)$. More qualitatively, we conclude that if $\gamma_B B \ll |\omega_B - \omega_A + 2\Gamma|$, then qubit B will not appreciably change when qubit A is fixed in the state $|0\rangle_A$.

- Finally, if qubit A is fixed in the state $|1\rangle_A$, then the part of H_{rot} affecting qubit B is

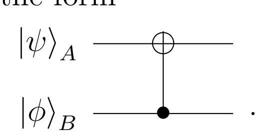
$$H_{\text{rot}}^{B,\,\text{if}\,|1\rangle_A} = \frac{\hbar}{2}\Big[(\omega_B - \omega_A)\sigma_z^B + \gamma_B B \sigma_x^B\Big].$$

  to precess around an axis in the xz-plane of its Bloch sphere that is at an angle

$$\chi = \arcsin\left(\frac{\gamma_B B}{\sqrt{(\omega_B - \omega_A)^2 + \gamma_B^2 B^2}}\right)$$

  clockwise from the z-axis. Thus, if qubit A is fixed in state $|1\rangle_A$ and qubit B begins along the z-axis then qubit B will always be in a state within 2χ of the z-axis. That is, for all t, $\langle\psi_B(0)|\psi_B(t)\rangle \geq \cos(2\chi)$. More qualitatively, we conclude that if $\gamma_B B \ll |\omega_B - \omega_A|$, then qubit B will not appreciably change when qubit A is fixed in the state $|1\rangle_A$.

So, in conclusion, H_{rot} applied for $t = \pi/(\gamma_A B)$ enacts unitary dynamics in the rotating frame approximating a CNOT of the form



The criteria for this approximation to be accurate are

- $\gamma_A B \ll 2\Gamma$, and
- $\gamma_B B \ll |\omega_B - \omega_A|$.

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2) Compare the effect of the sequence of rotations

$$e^{i\pi\sigma_y^A/4}\, e^{-i\pi(\sigma_z^A\otimes\sigma_z^B)/4}\, e^{-i\pi\sigma_x^A/4}$$

to that of a standard CNOT gate.

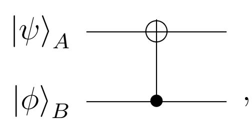
[*Hint*: The formula $e^{ikM} = (\cos k)\mathbb{I} + i(\sin k)M$ holds for any matrix M such that $M^2 = \mathbb{I}$, not just 2×2 matrices.]

Solution:

Taking the hint, we plug 'n' chug through the matrix arithmetic.

$$\begin{aligned}
&e^{i\pi\sigma_y^A/4}\, e^{-i\pi(\sigma_z^A\otimes\sigma_z^B)/4}\, e^{-i\pi\sigma_x^A/4} = \\
&= [\cos(\pi/4)\mathbb{I} + i\sin(\pi/4)\sigma_y^A][\cos(\pi/4)\mathbb{I} - i\sin(\pi/4)(\sigma_z^A\otimes\sigma_z^B)][\cos(\pi/4)\mathbb{I} - i\sin(\pi/4)\sigma_x^A] \\
&= \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1-i & 0 & 0 & 0 \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & 1+i & 0 \\ 0 & 0 & 0 & 1-i \end{pmatrix}\begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \\ -i & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \end{pmatrix} \\
&= \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1-i & 0 & -1-i & 0 \\ 0 & 1+i & 0 & 1-i \\ 1-i & 0 & 1+i & 0 \\ 0 & -1-i & 0 & 1-i \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\pi/4} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\pi/4} \\ 0 & 0 & e^{i\pi/4} & 0 \\ 0 & -e^{i\pi/4} & 0 & 0 \end{pmatrix}
\end{aligned}$$

In comparison, a CNOT of the form



has a matrix representation in the product Z basis of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

Thus, in terms of amplitudes only, the sequence of rotations is equivalent to a CNOT with qubit B as the control bit. However, there are nontrivial relative phases.

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3) Consider the 3 spin Hamiltonian

$$H = -\frac{\hbar}{2}\Big[\omega_A \sigma_z^A + \omega_B \sigma_z^B + \omega_C \sigma_z^C + \Gamma_{AB}(\sigma_z^A \otimes \sigma_z^B) + \Gamma_{AC}(\sigma_z^A \otimes \sigma_z^C) + \Gamma_{BC}(\sigma_z^B \otimes \sigma_z^C)\Big].$$

The presence of the interactions splits each of the individual NMR peaks ("singlets") at ω_A, ω_B, and ω_C that would occur in the noninteracting case into 4 peaks (a "doublet of doublets"). Derive formulas for the size of splittings of the ω_A peak in terms of Γ_{AB}, Γ_{AC}, and/or Γ_{BC}.

Solution:

All the terms in this Hamiltonian are diagonalized in the Pauli z-product basis. Thus, it is straightforward to find its eigenvalues and eigenvectors

$$\begin{aligned}
H|000\rangle &= -\frac{\hbar}{2}\big(+\omega_A + \omega_B + \omega_C + \Gamma_{AB} + \Gamma_{AC} + \Gamma_{BC}\big)|000\rangle = \hbar\omega_{000}|000\rangle \\
H|001\rangle &= -\frac{\hbar}{2}\big(+\omega_A + \omega_B - \omega_C + \Gamma_{AB} - \Gamma_{AC} - \Gamma_{BC}\big)|001\rangle = \hbar\omega_{001}|001\rangle \\
H|010\rangle &= -\frac{\hbar}{2}\big(+\omega_A - \omega_B + \omega_C - \Gamma_{AB} + \Gamma_{AC} - \Gamma_{BC}\big)|010\rangle = \hbar\omega_{010}|010\rangle \\
H|011\rangle &= -\frac{\hbar}{2}\big(+\omega_A - \omega_B - \omega_C - \Gamma_{AB} - \Gamma_{AC} + \Gamma_{BC}\big)|011\rangle = \hbar\omega_{011}|011\rangle \\
H|100\rangle &= -\frac{\hbar}{2}\big(-\omega_A + \omega_B + \omega_C - \Gamma_{AB} - \Gamma_{AC} + \Gamma_{BC}\big)|100\rangle = \hbar\omega_{100}|100\rangle \\
H|101\rangle &= -\frac{\hbar}{2}\big(-\omega_A + \omega_B - \omega_C - \Gamma_{AB} + \Gamma_{AC} - \Gamma_{BC}\big)|101\rangle = \hbar\omega_{101}|101\rangle \\
H|110\rangle &= -\frac{\hbar}{2}\big(-\omega_A - \omega_B + \omega_C + \Gamma_{AB} - \Gamma_{AC} - \Gamma_{BC}\big)|110\rangle = \hbar\omega_{110}|110\rangle \\
H|111\rangle &= -\frac{\hbar}{2}\big(-\omega_A - \omega_B - \omega_C + \Gamma_{AB} + \Gamma_{AC} + \Gamma_{BC}\big)|111\rangle = \hbar\omega_{111}|111\rangle
\end{aligned}$$

From the above equations, it can be seen that in the noninteracting case (that is, $\Gamma_{AB} = \Gamma_{AC} = \Gamma_{BC} = 0$) radiation at a frequency ω_A would be resonant with all four transitions of the form $|0bc\rangle \leftrightarrow |1bc\rangle$ where $b, c \in \{0, 1\}$. In the presence of the interactions though, these four transitions generically have distinct resonant frequencies

$$\begin{aligned}
|000\rangle \leftrightarrow |100\rangle : &\qquad \omega_{100} - \omega_{000} = \omega_A + \Gamma_{AB} + \Gamma_{AC} \\
|001\rangle \leftrightarrow |101\rangle : &\qquad \omega_{101} - \omega_{001} = \omega_A + \Gamma_{AB} - \Gamma_{AC} \\
|010\rangle \leftrightarrow |110\rangle : &\qquad \omega_{110} - \omega_{010} = \omega_A - \Gamma_{AB} + \Gamma_{AC} \\
|011\rangle \leftrightarrow |111\rangle : &\qquad \omega_{111} - \omega_{011} = \omega_A - \Gamma_{AB} - \Gamma_{AC}
\end{aligned}$$

The splittings of ω_A (that is, the differences between the resonant frequencies for transitions flipping qubit A in the interacting case and ω_A which is the resonant frequency for flipping qubit A in the noninteracting case) depend only on the couplings in which qubit A directly takes part and thus depend only on Γ_{AB} and Γ_{AC}. Specifically,

$$|0bc\rangle \leftrightarrow |1bc\rangle : \omega_{\text{splitting}} = (-1)^b \Gamma_{AB} + (-1)^c \Gamma_{AC}$$

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4) Consider the quantized simple harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2.$$

Define the annihilation operator a and creation operator $a^\dagger$,

$$a \equiv \frac{1}{\sqrt{2m\hbar\omega}} (m\omega x + ip); \quad a^\dagger \equiv \frac{1}{\sqrt{2m\hbar\omega}} (m\omega x - ip).$$

Verify the following four facts.

- $[a, a^\dagger] = \mathbb{I}$.

  Solution:

  By definition,

  $$[a, a^\dagger] = \frac{1}{2m\hbar\omega} [m\omega x + ip, m\omega x - ip].$$

  Now note the distributivity of commutators over addition

  $$[A + B, C + D] = [A, C] + [A, D] + [B, C] + [B, D]$$

  and note that, trivially, $[kA, lB] = kl[A, B]$ for scalars k and l and $[A, A] = 0$. With these 3 facts, the above equation for $[a, a^\dagger]$ simplifies to

  $$[a, a^\dagger] = \frac{i}{2\hbar} ([x, -p] + [p, x])$$

  Finally, since $[x, p] = i\hbar$, we prove the quoted result.

- $[a, a^\dagger a] = a$.

  Solution:

  As $[a, a^\dagger a] = [a, a^\dagger]a$, we need only invoke the just proved fact $[a, a^\dagger] = \mathbb{I}$ to prove the quoted result.

- $[a^\dagger, a^\dagger a] = -a^\dagger$.

  Solution:

  As $[a^\dagger, a^\dagger a] = -[a^\dagger a, a^\dagger] = -a^\dagger[a, a^\dagger]$, once again we need only invoke the fact $[a, a^\dagger] = \mathbb{I}$ to prove the quoted result.

- $H = \dfrac{\hbar\omega}{2}\left(aa^\dagger + a^\dagger a\right) = \hbar\omega\left(a^\dagger a + \dfrac{1}{2}\right)$.

  Solution:

  First, we invert the relationships giving a and $a^\dagger$ in terms of x and p

$$\begin{aligned} a &\equiv \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + ip); \quad a^\dagger \equiv \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x - ip) \\ &\Longrightarrow \quad x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right); \quad p = i\sqrt{\frac{m\hbar\omega}{2}}\left(a^\dagger - a\right) \end{aligned}$$

  Using these expressions, we rewrite H as follows

$$\begin{aligned} H &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \\ &= -\frac{\hbar\omega}{4}\left(a^\dagger - a\right)^2 + \frac{\hbar\omega}{4}\left(a + a^\dagger\right)^2 \\ &= \frac{\hbar\omega}{2}\left(aa^\dagger + a^\dagger a\right) \end{aligned}$$

  Finally, using the fact $[a, a^\dagger] = \mathbb{I}$, we see $aa^\dagger = \mathbb{I} + a^\dagger a$, and thus we conclude

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right).$$

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5) Let $|0\rangle$ denote the vacuum state for the quantized simple harmonic oscillator.

$$|0\rangle \equiv \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \int dx\; e^{-m\omega x^2/(2\hbar)}\,|x\rangle = \left(\frac{1}{\pi\hbar m\omega}\right)^{1/4} \int dp\; e^{-p^2/(2m\hbar\omega)}\,|p\rangle.$$

Verify that $a|0\rangle = 0$.

Solution:

First, a note of caution. Some might be tempted to argue very quickly that $a|0\rangle = 0$ because $a \equiv \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + ip)$ and because x and p are odd functions on the interval $(-\infty, \infty)$, whereas the Gaussian terms $e^{-m\omega x^2/(2\hbar)}$ and $e^{-p^2/(2m\hbar\omega)}$ that appear in the expressions for $|0\rangle$ are even functions. However, this is not a valid argument to show $a|0\rangle = 0$ because the integrals in the expressions for $|0\rangle$ are not simple integrals of scalar functions, but rather prescriptions for linear combinations of an infinite number of basis kets in an infinite-dimensional Hilbert space. Those Gaussian terms are weights for different kets $|x\rangle$ and $|p\rangle$. To show $a|0\rangle = 0$, we need to show that for all x and p, the weights for the respective kets $|x\rangle$ or $|p\rangle$ vanishes.

This can be straightforwardly accomplished by using the expression for $|0\rangle$ in the x-basis and the x-basis representation for momentum, $p = -i\hbar\partial/\partial x$.

$$\begin{aligned} a|0\rangle &= \frac{1}{\sqrt{2m\hbar\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\int dx\,(m\omega x + ip)\; e^{-m\omega x^2/(2\hbar)}\,|x\rangle \\ &= \frac{1}{\sqrt{2m\hbar\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\int dx\left(m\omega x e^{-m\omega x^2/(2\hbar)} - i^2\hbar\frac{\partial}{\partial x}\left[e^{-m\omega x^2/(2\hbar)}\right]\right)|x\rangle \\ &= \frac{1}{\sqrt{2m\hbar\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\int dx\left(m\omega x e^{-m\omega x^2/(2\hbar)} + \hbar\left(\frac{-m\omega}{2\hbar}\right)(2x)\left[e^{-m\omega x^2/(2\hbar)}\right]\right)|x\rangle \\ &= 0 \end{aligned}$$

as the integrand simply vanishes.

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6) Define the nth excited state of the quantized simple harmonic oscillator to be $|n\rangle \equiv c_n(a^\dagger)^n|0\rangle$. Verify the following four facts.

- $a|n\rangle = \sqrt{n}|n-1\rangle$.
- $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.
- $H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$.
- $c_n = \dfrac{1}{\sqrt{n!}}$ is the proper normalization constant.

Solution:

We shall first establish the final fact: $c_n = 1/\sqrt{n!}$ is the correct normalization. To do this, we need to establish what a and $a^\dagger$ to the unnormalized nth excited state $|n\rangle_{\rm un} = (a^\dagger)^n|0\rangle$.

We immediately see that

$$a^\dagger|n\rangle_{\rm un} = (a^\dagger)^{n+1}|0\rangle = |n+1\rangle_{\rm un}.$$

Using $[a, a^\dagger] = \mathbb{I}$, we calculate

$$\begin{aligned} a|n\rangle_{\rm un} &= a(a^\dagger)^n|0\rangle \\ &= (aa^\dagger)(a^\dagger)^{n-1}|0\rangle \\ &= (\mathbb{I} + a^\dagger a)|n-1\rangle_{\rm un}. \end{aligned}$$

Iteratively applying this fact

$$
\begin{aligned}
a|n\rangle_{\text{un}} &= (\mathbb{I} + a^\dagger a)|n-1\rangle_{\text{un}} \\
&= |n-1\rangle_{\text{un}} + a^\dagger(\mathbb{I} + a^\dagger a)|n-2\rangle_{\text{un}} \\
&= 2|n-1\rangle_{\text{un}} + (a^\dagger)^2 a|n-2\rangle_{\text{un}} \\
&\vdots \\
&= k|n-1\rangle_{\text{un}} + (a^\dagger)^k a|n-k\rangle_{\text{un}} \\
&\vdots \\
&= n|n-1\rangle_{\text{un}} + (a^\dagger)^n a|0\rangle \\
&= n|n-1\rangle_{\text{un}} \text{ since } a|0\rangle = 0.
\end{aligned}
$$

Combining these two facts yields

$$a^\dagger a|n\rangle_{\text{un}} = n|n\rangle_{\text{un}}.$$

Correct normalization requires choosing c_n so that

$$\langle n|n\rangle \equiv |c_n|^2 \; {}_{\text{un}}\langle n|n\rangle_{\text{un}} = 1.$$

We now note the following recursion relation

$$
\begin{aligned}
{}_{\text{un}}\langle n|n\rangle_{\text{un}} &\equiv \langle 0|a^n(a^\dagger)^n|0\rangle \\
&= {}_{\text{un}}\langle n-1|aa^\dagger|n-1\rangle_{\text{un}} \\
&= {}_{\text{un}}\langle n-1|\mathbb{I} + a^\dagger a|n-1\rangle_{\text{un}} \\
&= n \cdot {}_{\text{un}}\langle n-1|n-1\rangle_{\text{un}}.
\end{aligned}
$$

Therefore, iteratively applying the recursion relation n times to ${}_{\text{un}}\langle n|n\rangle_{\text{un}}$ yields the quoted normalization

$$|c_n|^2 = \frac{1}{\sqrt{n!}}.$$

Using this with our formulas for $a|n\rangle_{\text{un}}$, $a^\dagger|n\rangle_{\text{un}}$, and $a^\dagger a|n\rangle_{\text{un}}$ establishes the other 3 facts we are to prove.

- $a|n\rangle = a\left(\frac{1}{\sqrt{n!}}|n\rangle_{\text{un}}\right) = \frac{n}{\sqrt{n!}}|n-1\rangle_{\text{un}} = \sqrt{n}|n-1\rangle$
- $a^\dagger|n\rangle = a^\dagger\left(\frac{1}{\sqrt{n!}}|n\rangle_{\text{un}}\right) = \frac{1}{\sqrt{n!}}|n+1\rangle_{\text{un}} = \sqrt{n+1}|n+1\rangle$
- $H|n\rangle = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)\left(\frac{1}{\sqrt{n!}}|n\rangle_{\text{un}}\right) = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$

7) Consider the following form of the Jaynes-Cummings Hamiltonian,

$$H = \hbar(C + D) + \frac{\hbar\omega}{2},$$

where $C \equiv \omega\left(a^\dagger a + \frac{1}{2}\sigma_z\right)$ and $D \equiv \kappa(a^\dagger\sigma_- + a\sigma_+) - \frac{\omega - \omega_0}{2}\sigma_z$. Verify that $[C, D] = 0$.

Solution:

As a and $a^\dagger$ both commute with σ_z and $\sigma_\pm$, the only nontrivial terms once one distributes out the commutators are

$$[C, D] = \omega\kappa\Big([a^\dagger a, a^\dagger]\sigma_- + [a^\dagger a, a]\sigma_+\Big) + \frac{\omega\kappa}{2}\Big(a^\dagger[\sigma_z, \sigma_-] + a[\sigma_z, \sigma_+]\Big).$$

From the results of Problem 4, we see $[a^\dagger a, a^\dagger] = a^\dagger$ and $[a^\dagger a, a] = -a$. As for the $[\sigma_z, \sigma_\pm]$ terms, recall that $[\sigma_x, \sigma_y] = 2i\sigma_z$ (and cyclic permutations thereof) and that $\sigma_\pm \equiv \frac{1}{2}(\sigma_x \pm i\sigma_y)$. Hence,

$$[\sigma_z, \sigma_\pm] = \frac{1}{2}\Big([\sigma_z, \sigma_x] \pm i[\sigma_z, \sigma_y]\Big) = \frac{1}{2}\Big(2i\sigma_y \pm i(-2i\sigma_x)\Big) = \pm\sigma_x + i\sigma_y = \pm 2\sigma_\pm.$$

We thus conclude

$$[C, D] = \omega\kappa\Big(a^\dagger\sigma_- - a\sigma_+\Big) + \frac{\omega\kappa}{2}\Big(a^\dagger(-2\sigma_-) + a(2\sigma_+)\Big) = 0.$$