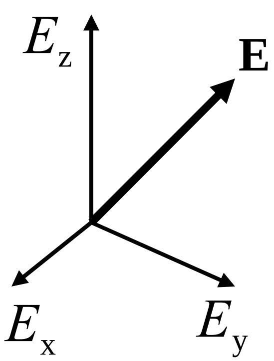
Elementary waves: plane, spherical

The EM vector wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\mathbf{E} = E_{\mathrm{x}}\hat{\mathbf{x}} + E_{\mathrm{y}}\hat{\mathbf{y}} + E_{\mathrm{z}}\hat{\mathbf{z}}$$

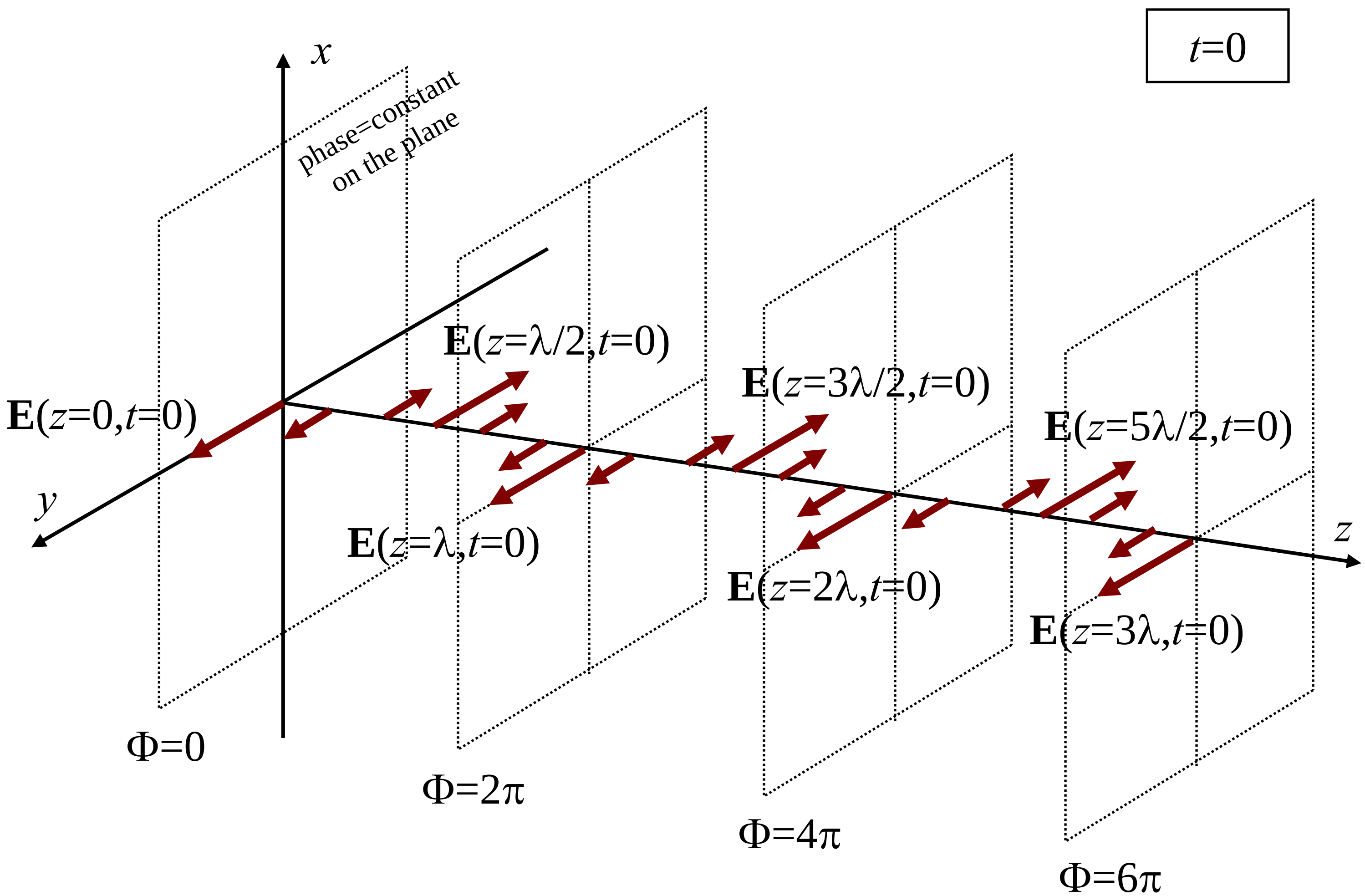


$$\frac{\partial^2 E_{\mathrm{x}}}{\partial x^2} + \frac{\partial^2 E_{\mathrm{x}}}{\partial y^2} + \frac{\partial^2 E_{\mathrm{x}}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_{\mathrm{x}}}{\partial t^2} = 0$$

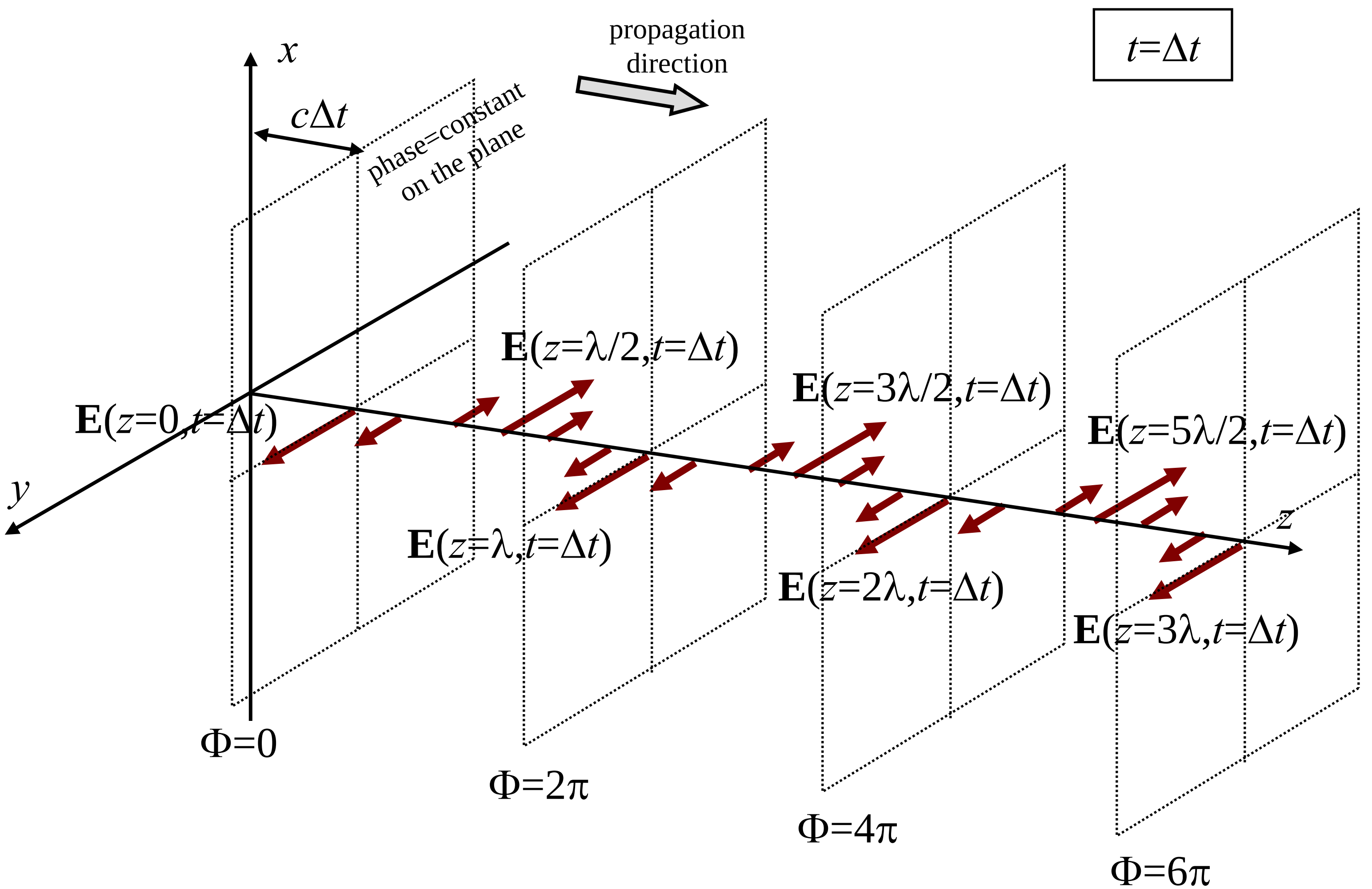
$$\frac{\partial^2 E_{\mathrm{y}}}{\partial x^2} + \frac{\partial^2 E_{\mathrm{y}}}{\partial y^2} + \frac{\partial^2 E_{\mathrm{y}}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_{\mathrm{y}}}{\partial t^2} = 0$$

$$\frac{\partial^2 E_{\mathrm{z}}}{\partial x^2} + \frac{\partial^2 E_{\mathrm{z}}}{\partial y^2} + \frac{\partial^2 E_{\mathrm{z}}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_{\mathrm{z}}}{\partial t^2} = 0$$

Harmonic solution in 3D: plane wave



Plane wave propagating



Complex representation of 3D waves

$$f(x,y,z,t) = A\cos\left(\frac{2\pi}{\lambda}\left(x\cos\alpha + y\cos\beta + z\cos\gamma\right) - \omega t - \phi_0\right)$$

$$f(x,y,z,t) = A\cos\left(k_x x + k_y y + k_z z - \omega t - \phi_0\right) \qquad k_x = \frac{2\pi}{\lambda}\cos\alpha,\ \text{etc.}$$

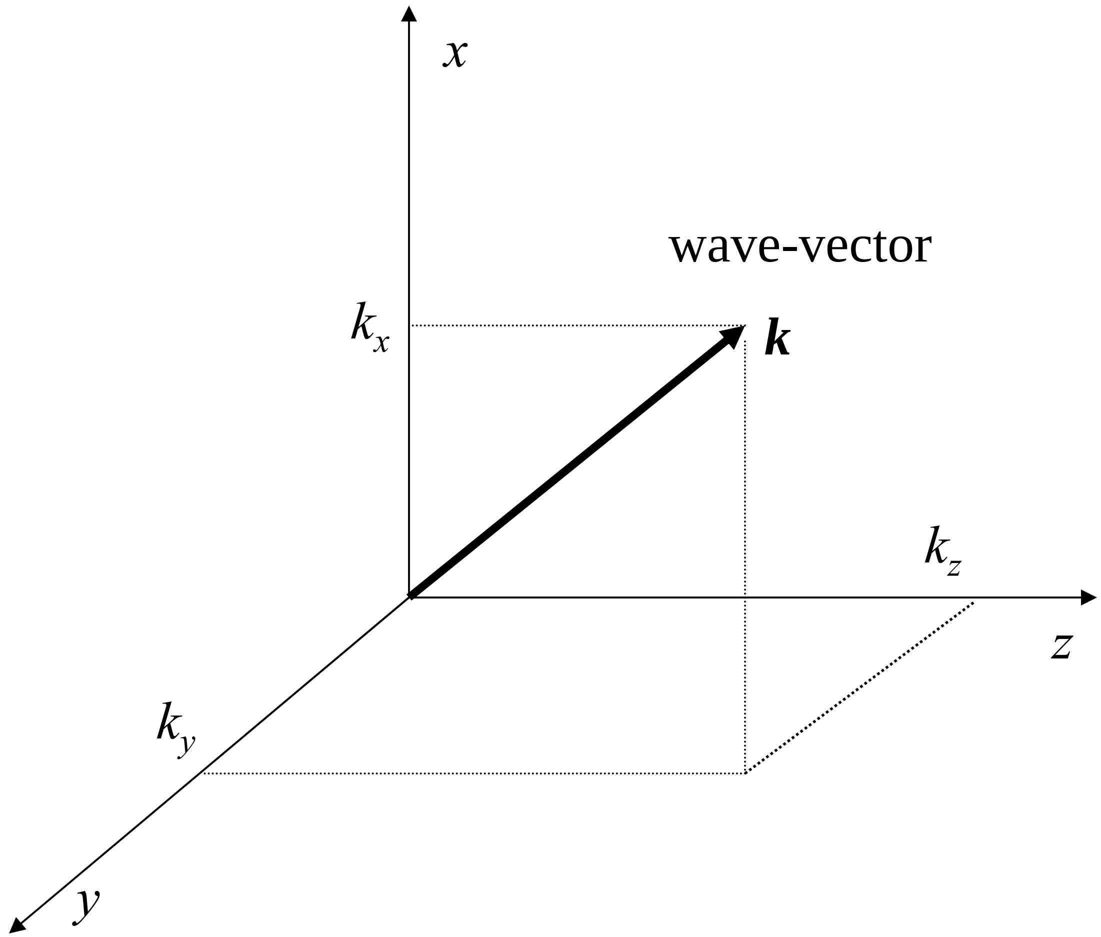
$$\hat{f}(x,y,z,t) = A\,\mathrm{e}^{i\left(k_x x + k_y y + k_z z - \omega t - \phi_0\right)} \quad \text{complex representation}$$

$$A\,\mathrm{e}^{-i\phi(x,y,z)} \quad \text{complex amplitude or "phasor"}$$

$$\text{where } \phi(x,y,z) \equiv k_x x + k_y y + k_z z - \phi_0$$

"Wavefront" : surface $\phi(x,y,z) = \text{const.}$

Plane wave



Plane wave

$$a(\mathbf{r}) = A_0\, \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

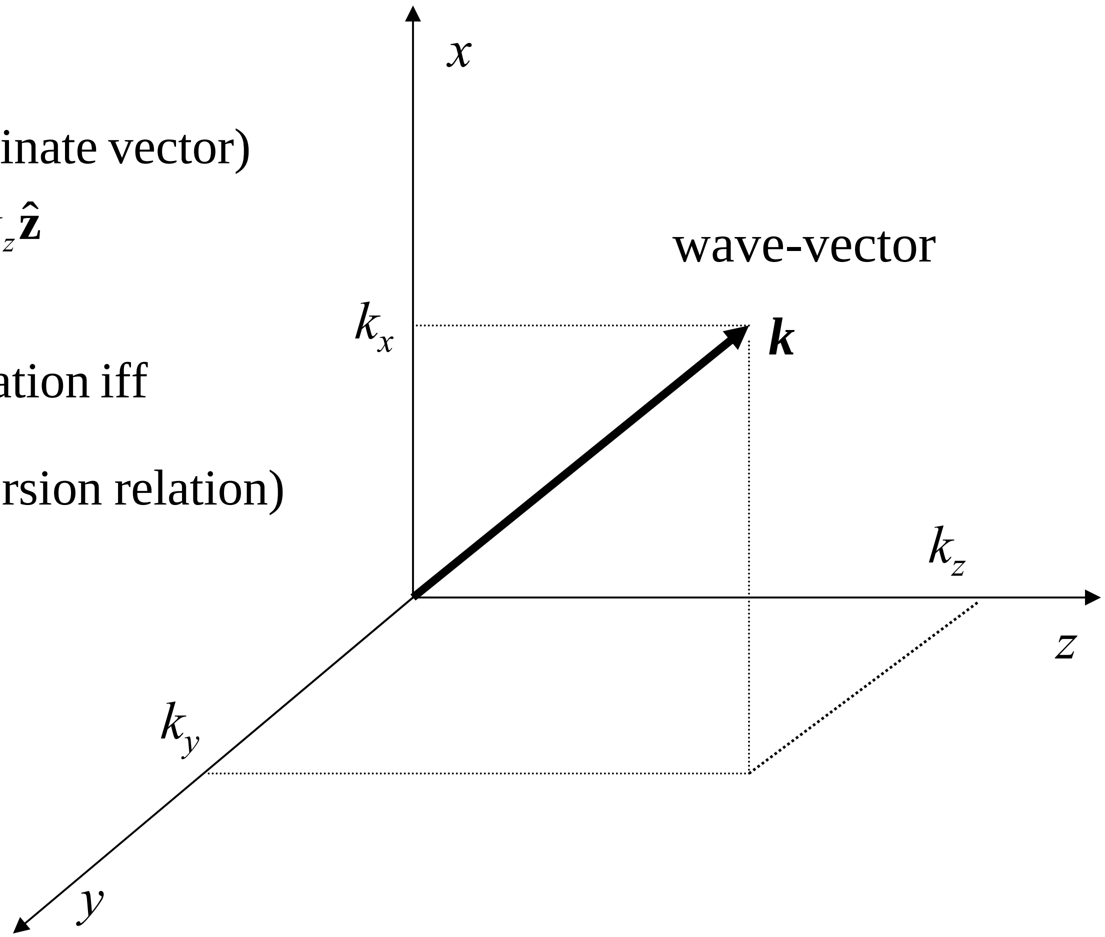
$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

(Cartesian coordinate vector)

$$\mathbf{k} = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} + k_z\hat{\mathbf{z}}$$

solves wave equation iff

$$|\mathbf{k}| = \frac{\omega}{c} \qquad \text{(dispersion relation)}$$



Plane wave

$$a(\mathbf{r}) = A_0\, \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

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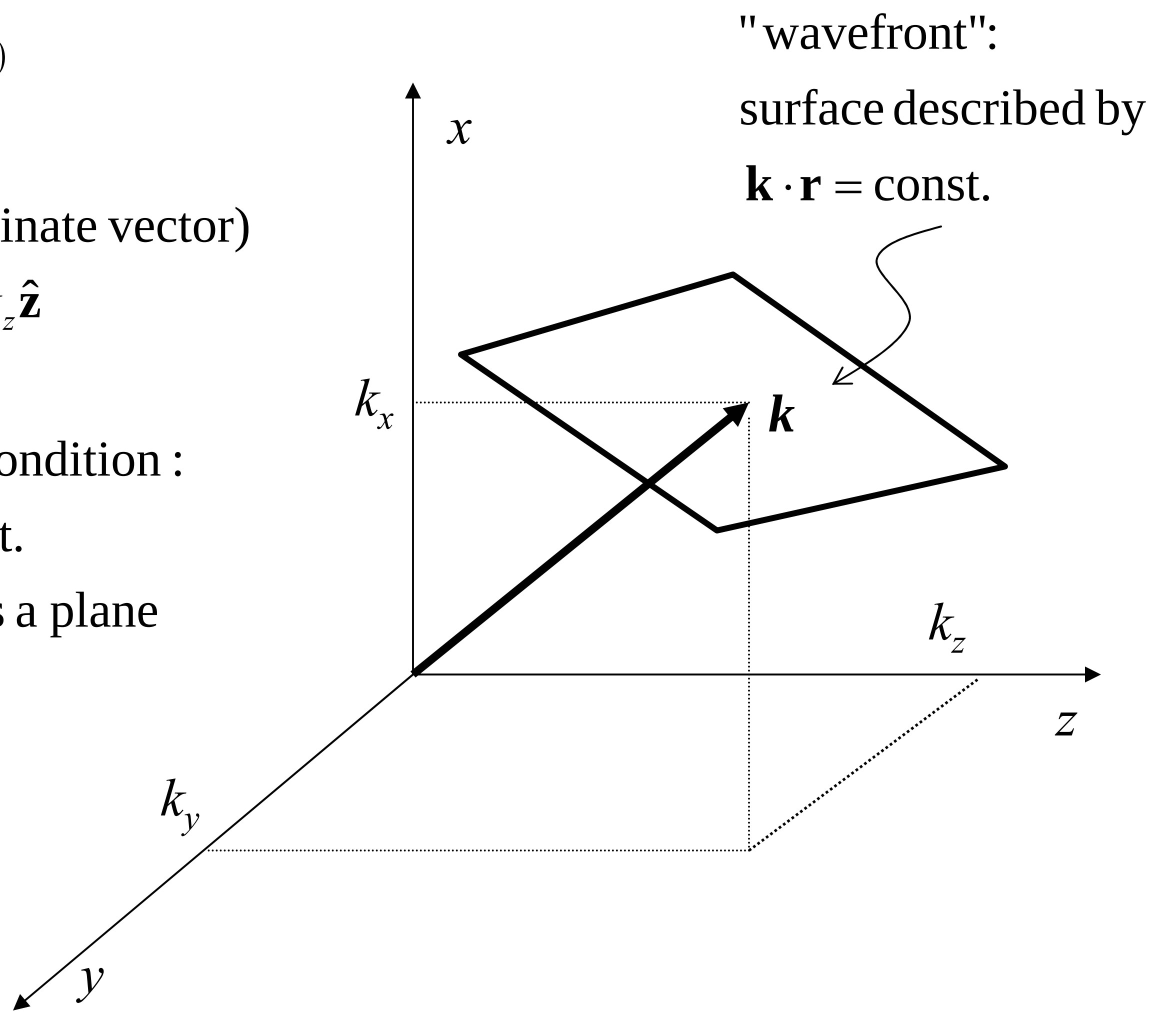
(Cartesian coordinate vector)

$$\mathbf{k} = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} + k_z\hat{\mathbf{z}}$$

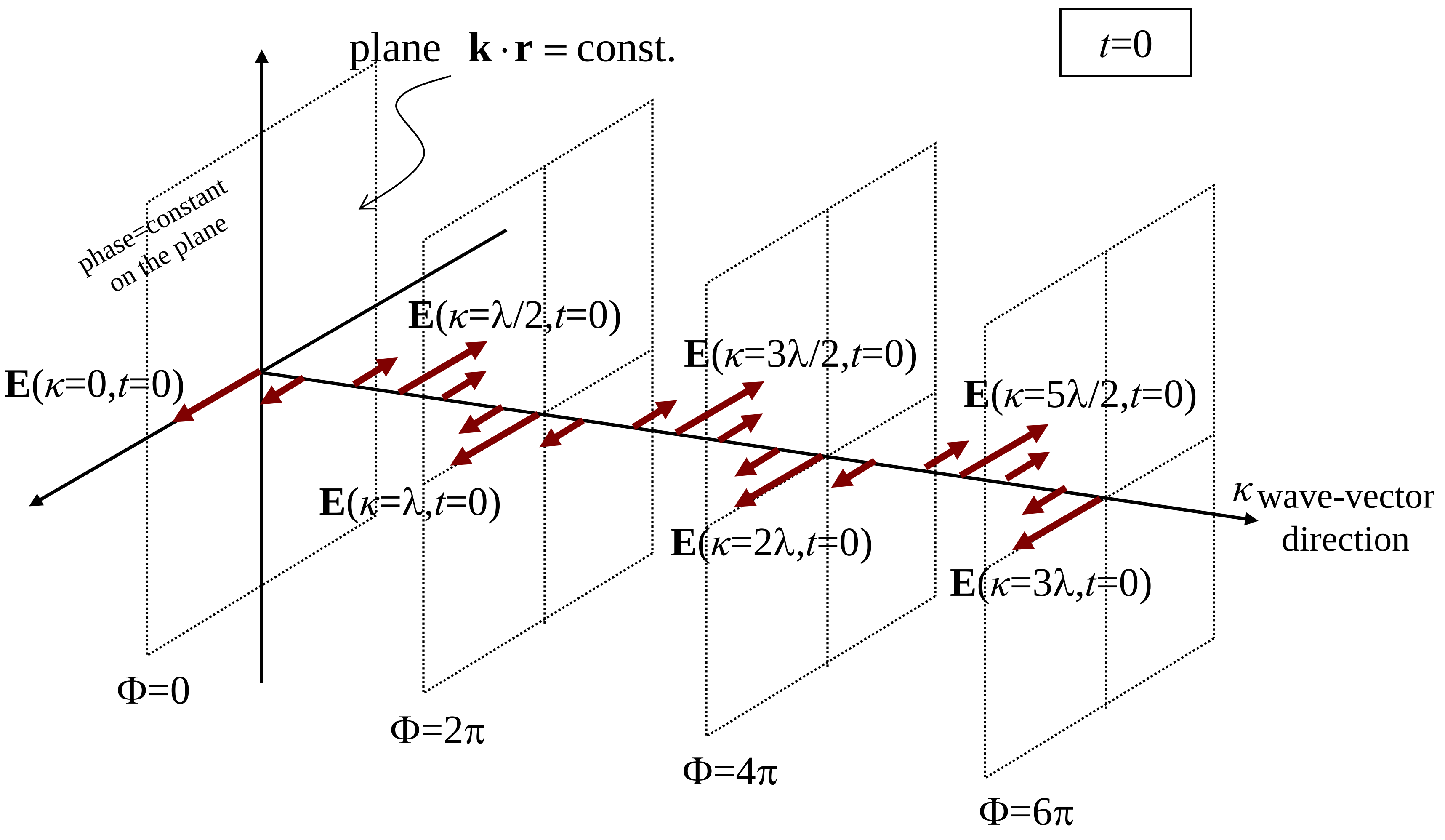
constant phase condition :

$$\mathbf{k}\cdot\mathbf{r} - \omega t = \text{const.}$$

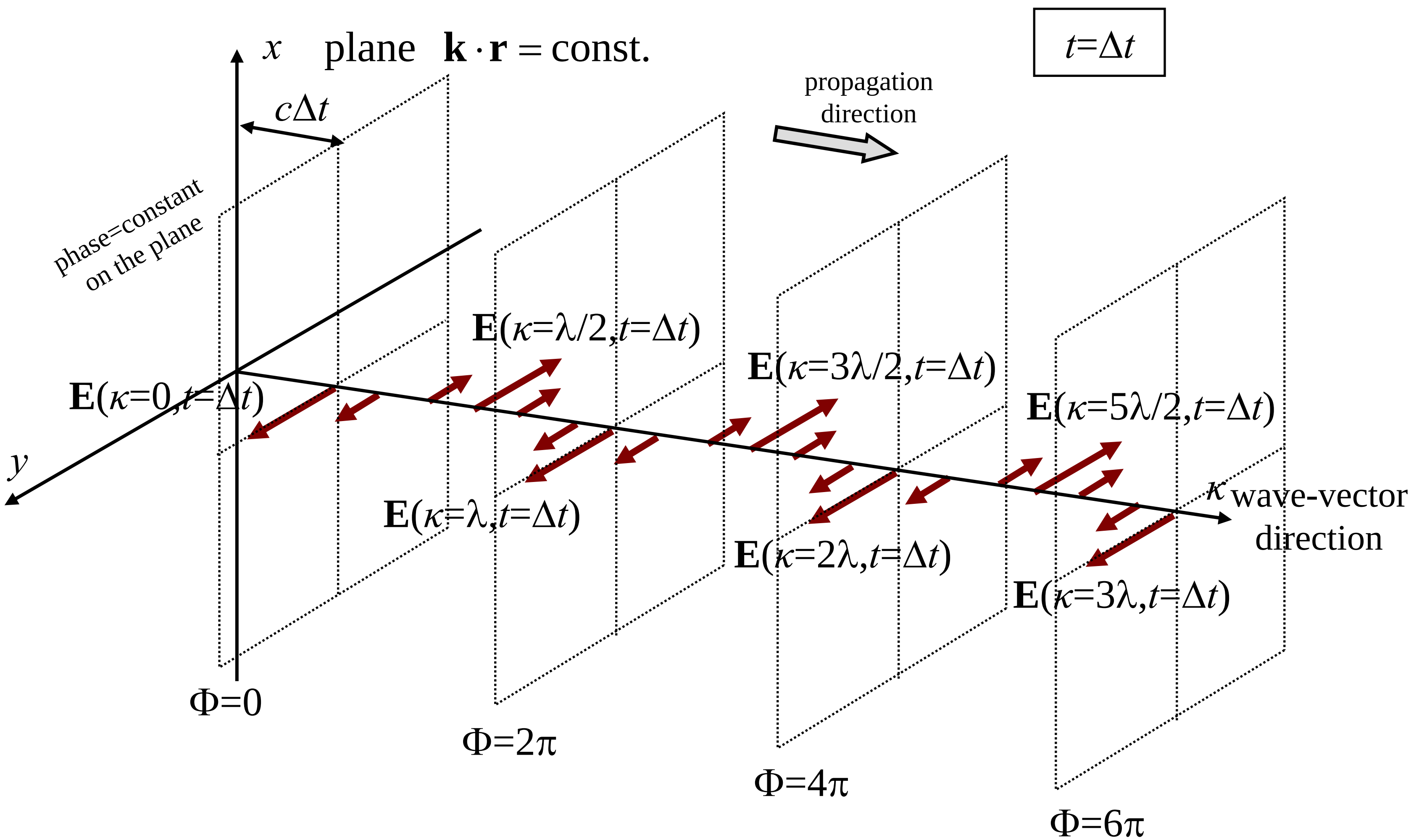
$\Rightarrow$ wave-front is a plane



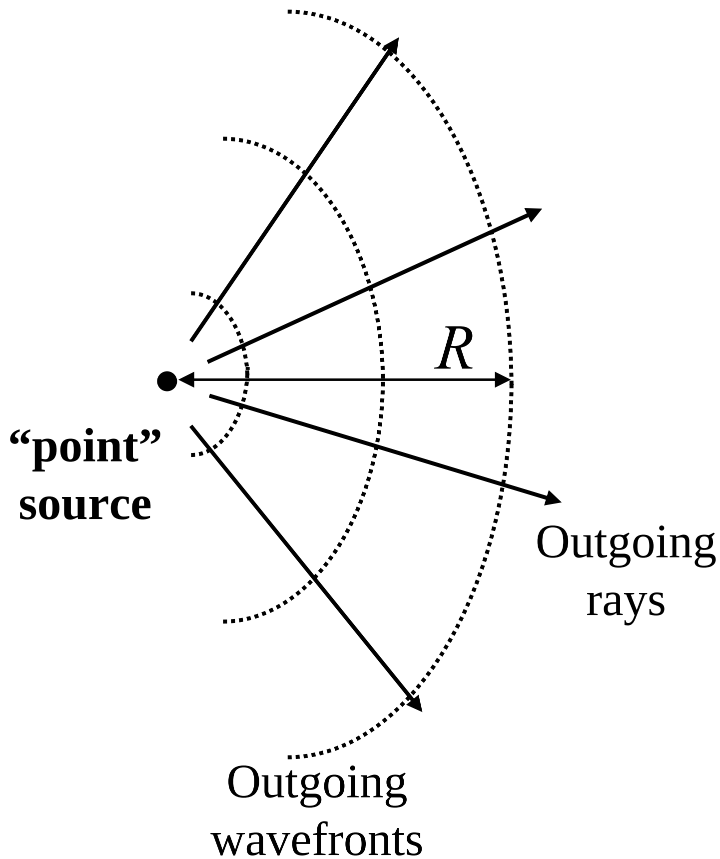
Plane wave propagating



Plane wave propagating



Spherical wave



equation of wavefront

$$kR - \omega t = \text{constant}$$

$$a(x,y,z,t) = A\,\frac{\cos\left(kR - \omega t + \pi/2\right)}{R}$$

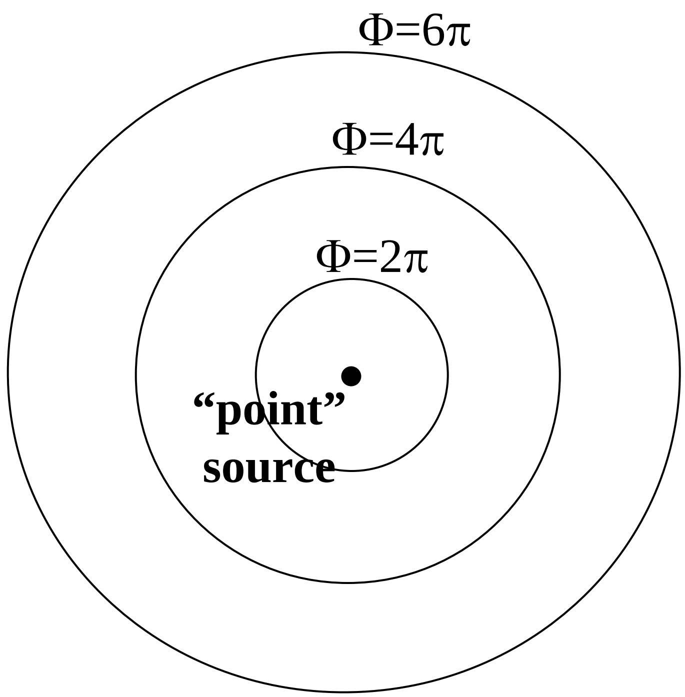
exponential notation

$$a(x,y,z,\cancel{t}) = A\,\frac{\exp\left\{\, i\left(kR - \cancel{\omega t}\right)\right\}}{iR}$$

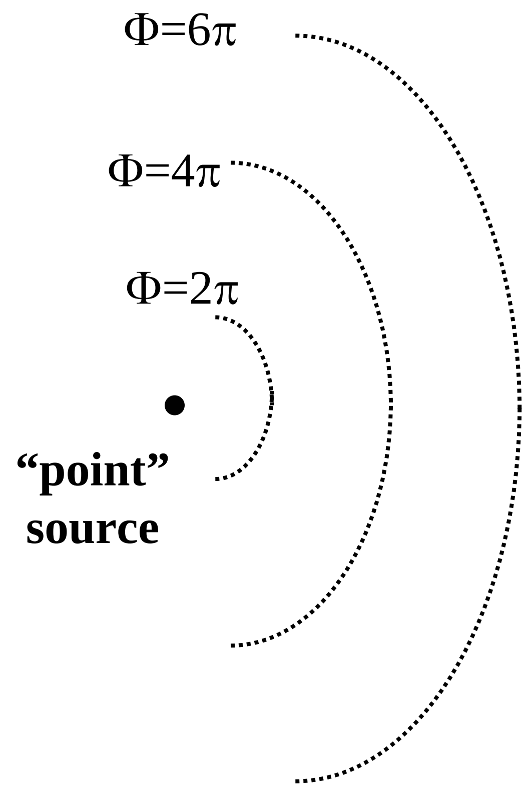
paraxial approximation

$$a(x,y,z) = \frac{A}{iR}\exp\left\{\, i2\pi\,\frac{z}{\lambda} + i\pi\,\frac{x^2+y^2}{\lambda z}\right\}$$

Spherical wave

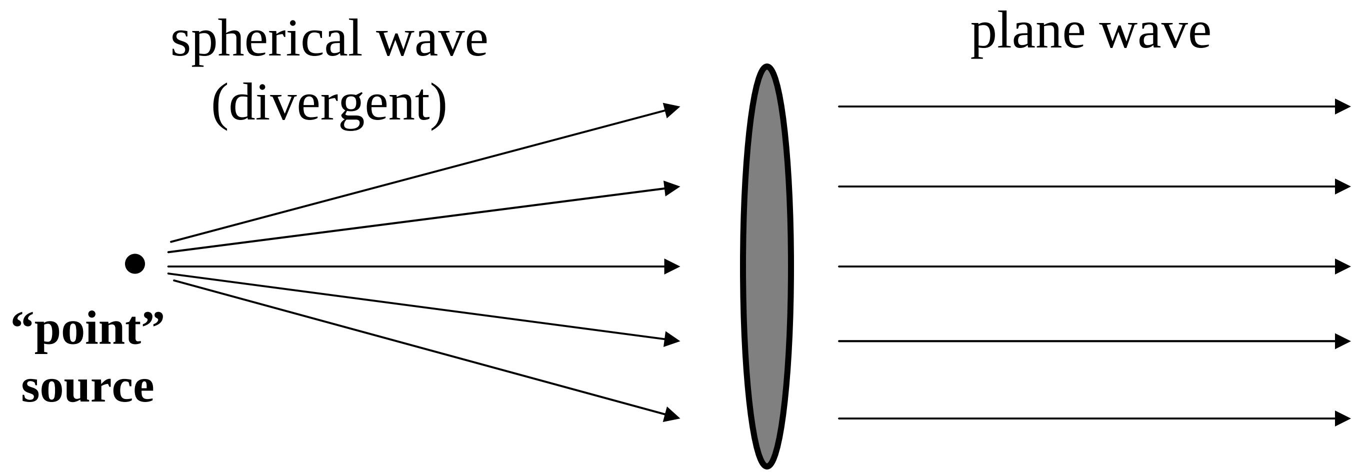


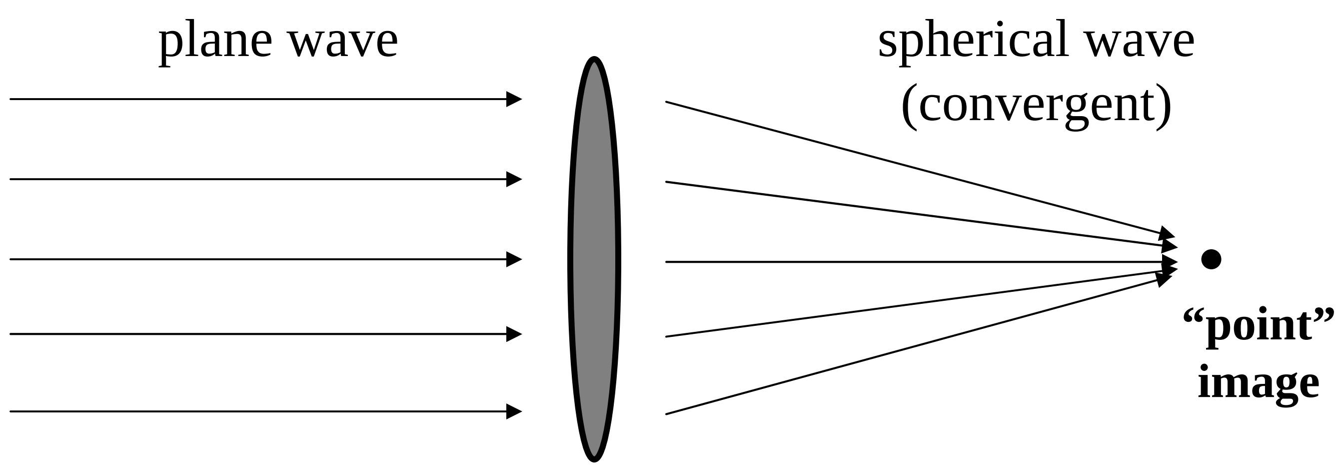
exact



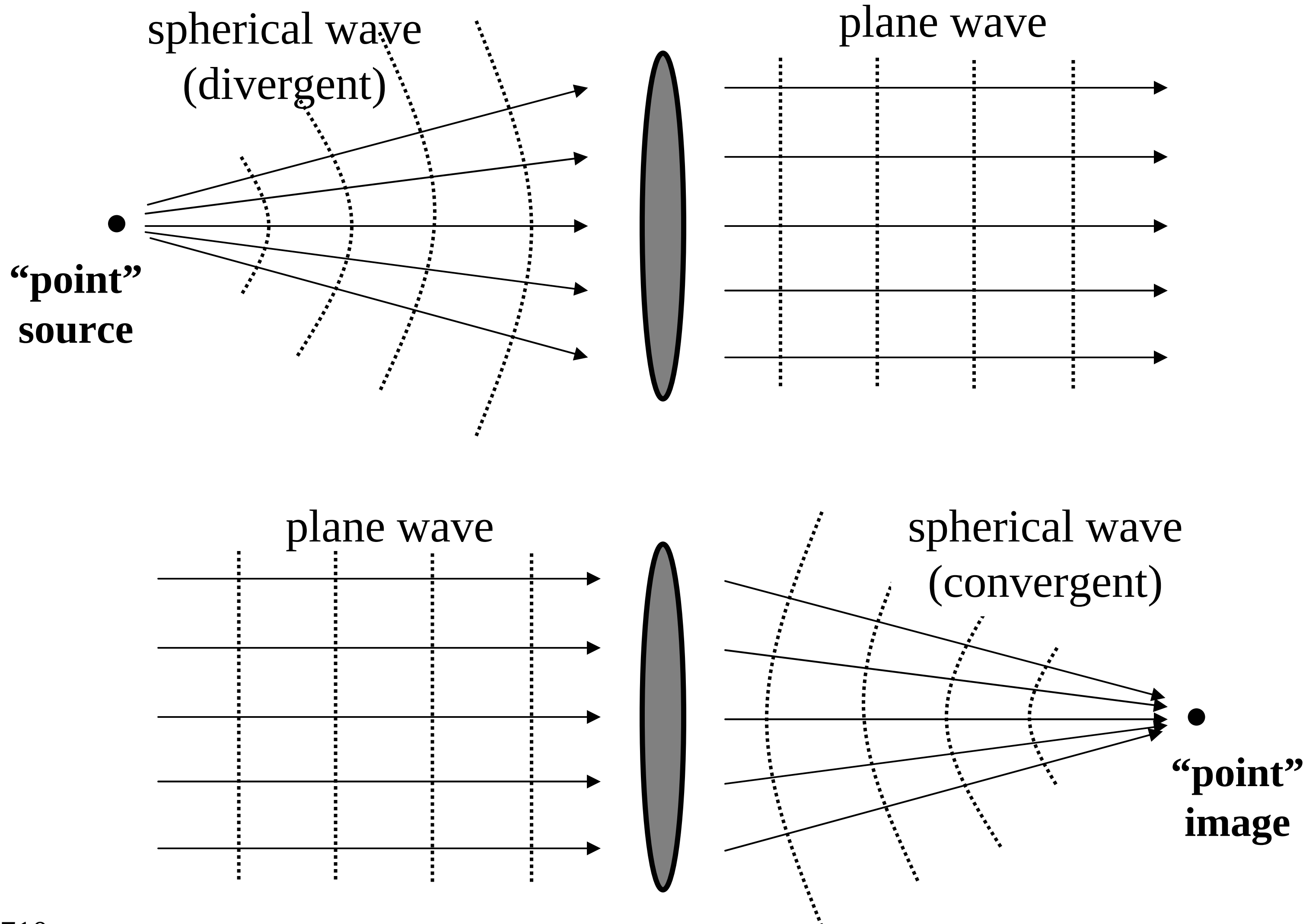
paraxial approximation/
/Gaussian beams

The role of lenses





The role of lenses



Polarization

Propagation and polarization

In isotropic media (e.g. free space, amorphous glass, etc.)

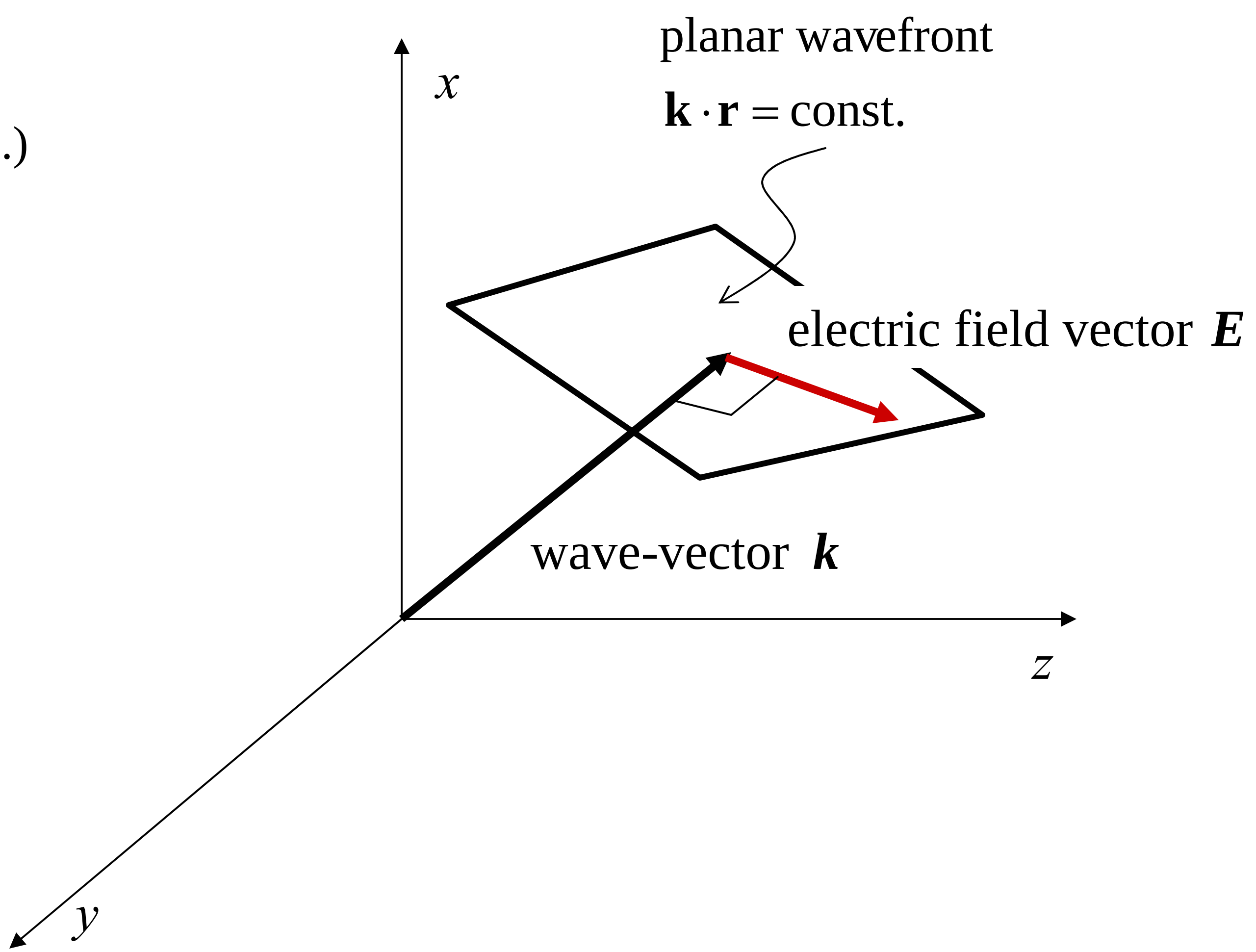
$\mathbf{k} \cdot \mathbf{E} = 0$

i.e. $\mathbf{k} \perp \mathbf{E}$

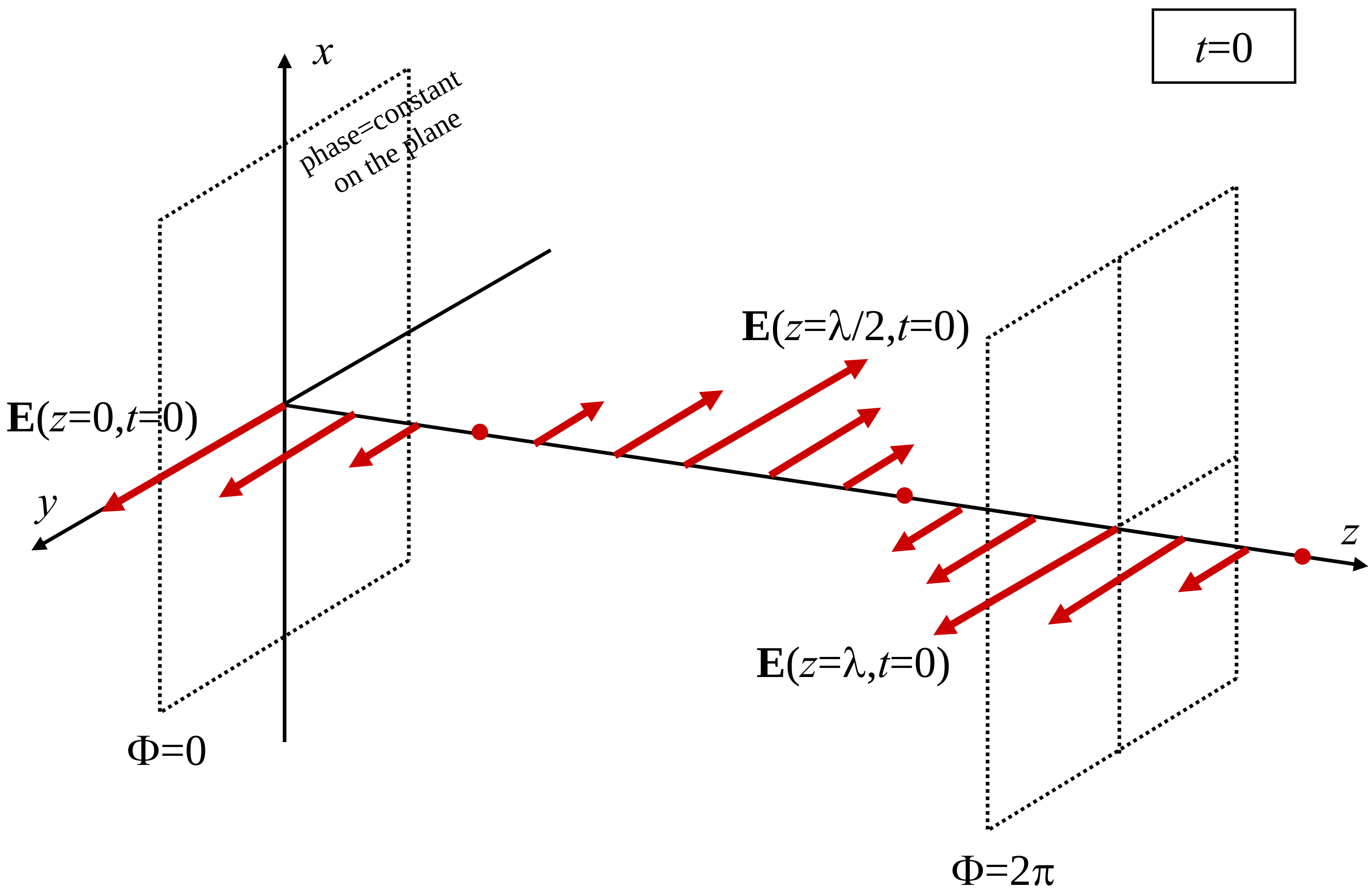
More generally,

$\mathbf{k} \cdot \mathbf{D} = 0$

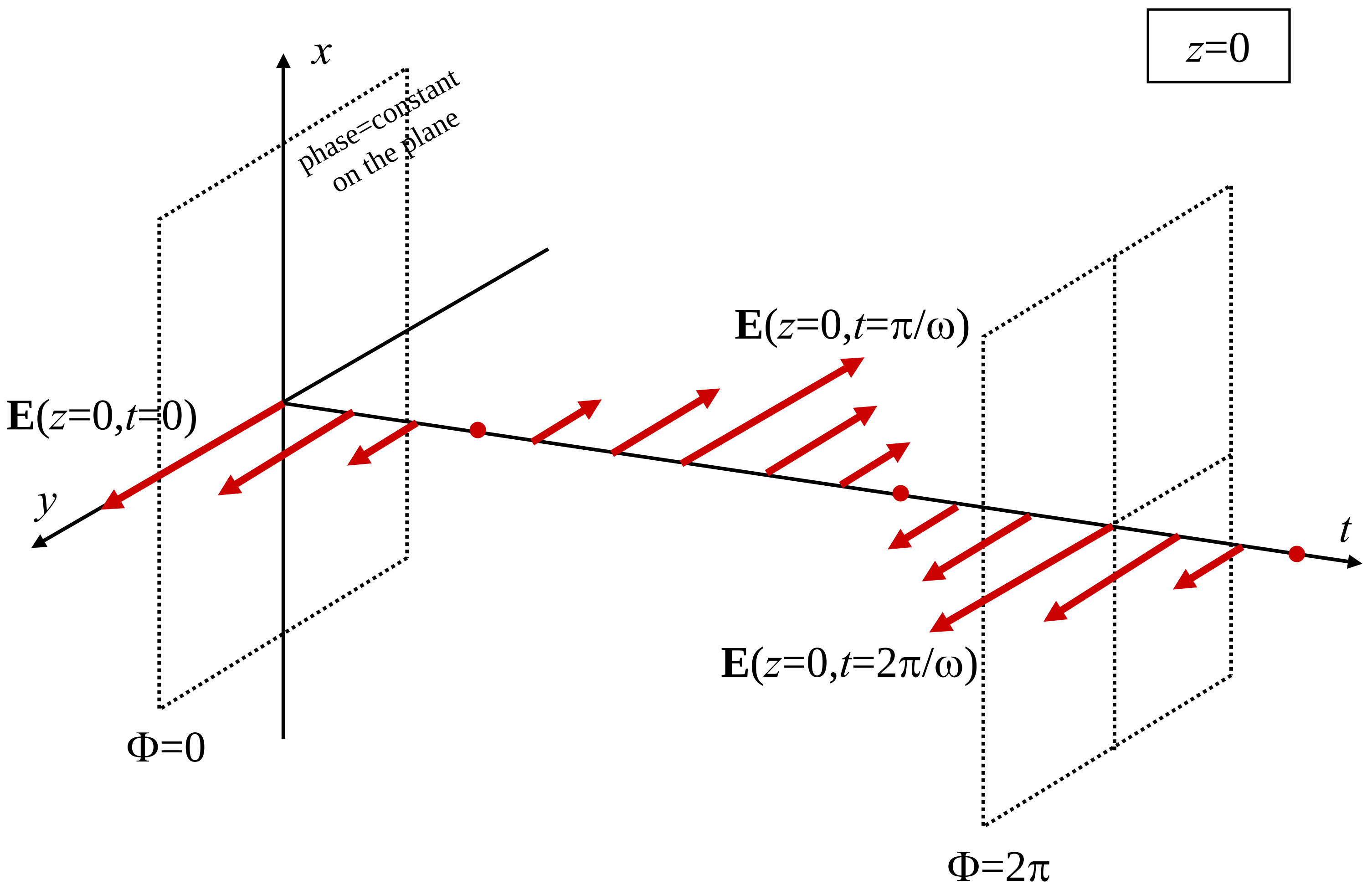
(reminder: in anisotropic media, e.g. crystals, one could have **E** not parallel to **D**)



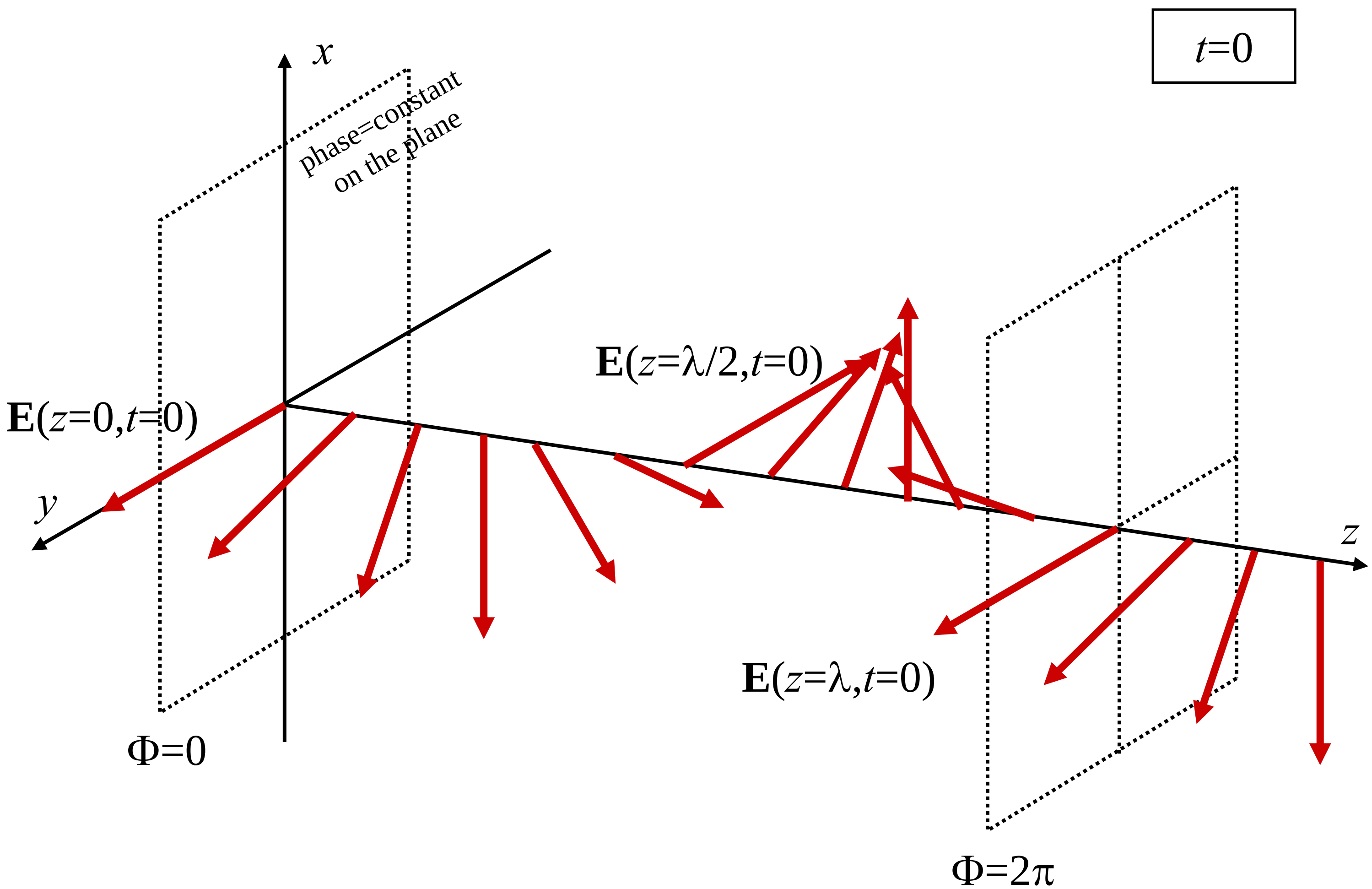
Linear polarization (frozen time)



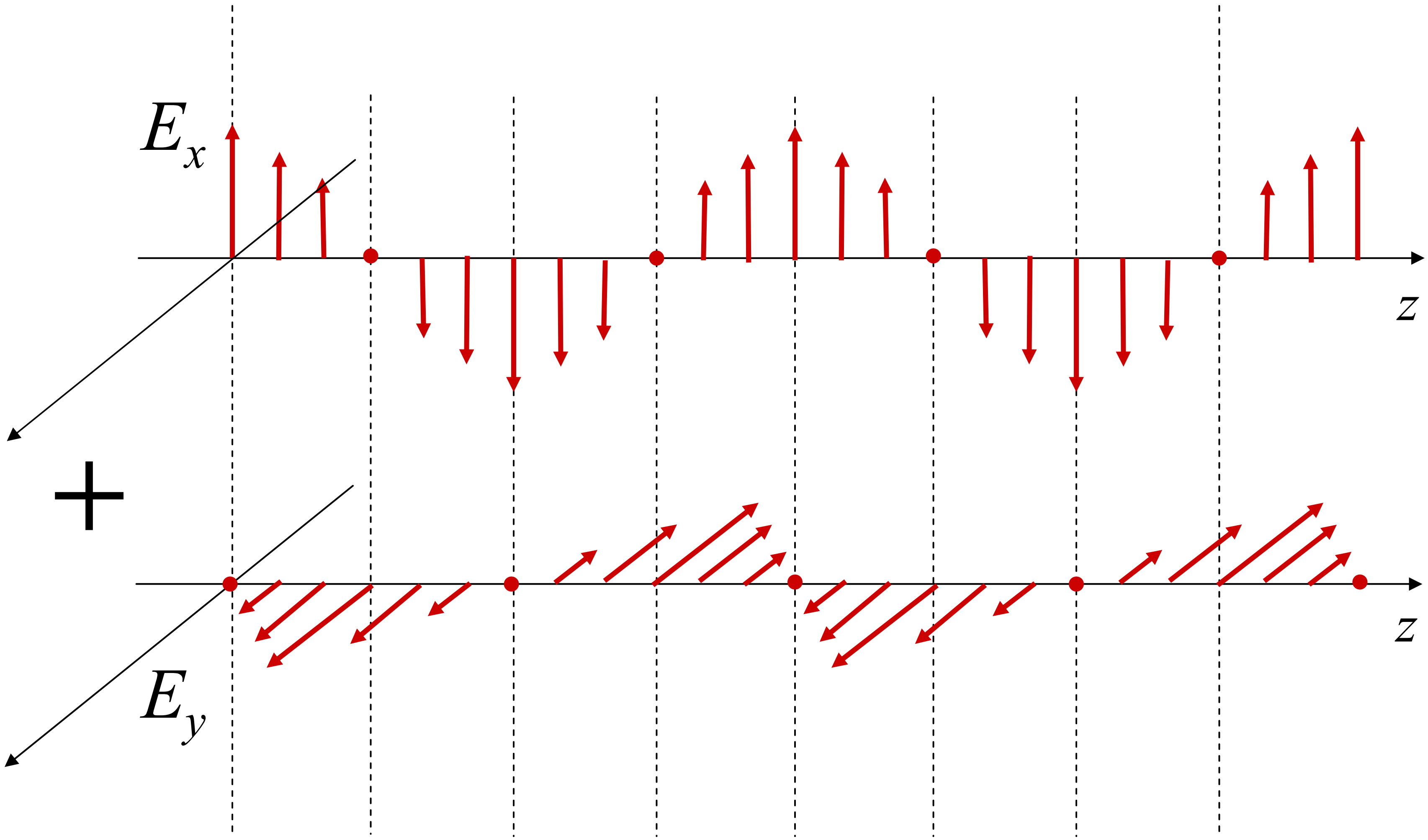
Linear polarization (fixed space)



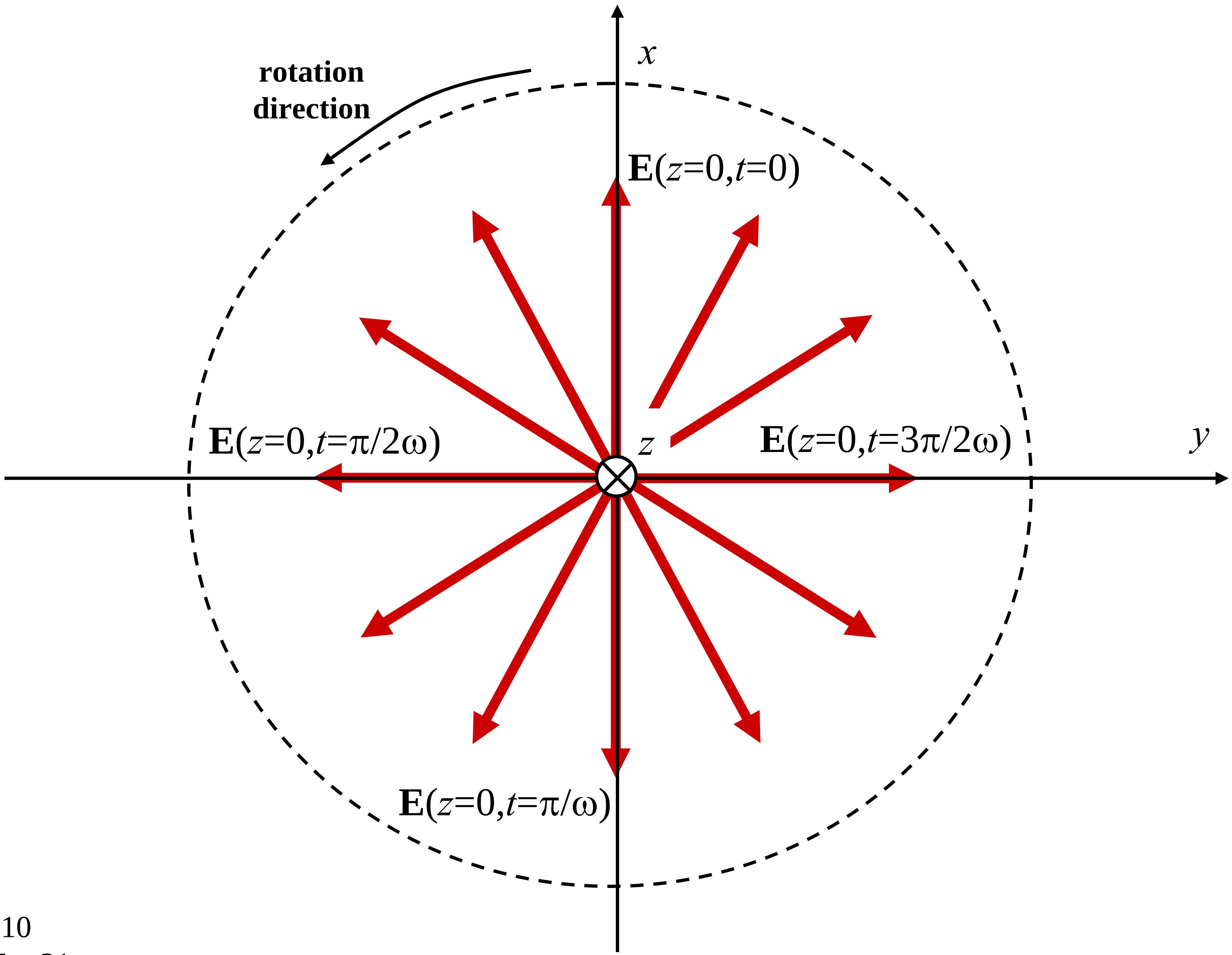
Circular polarization (frozen time)



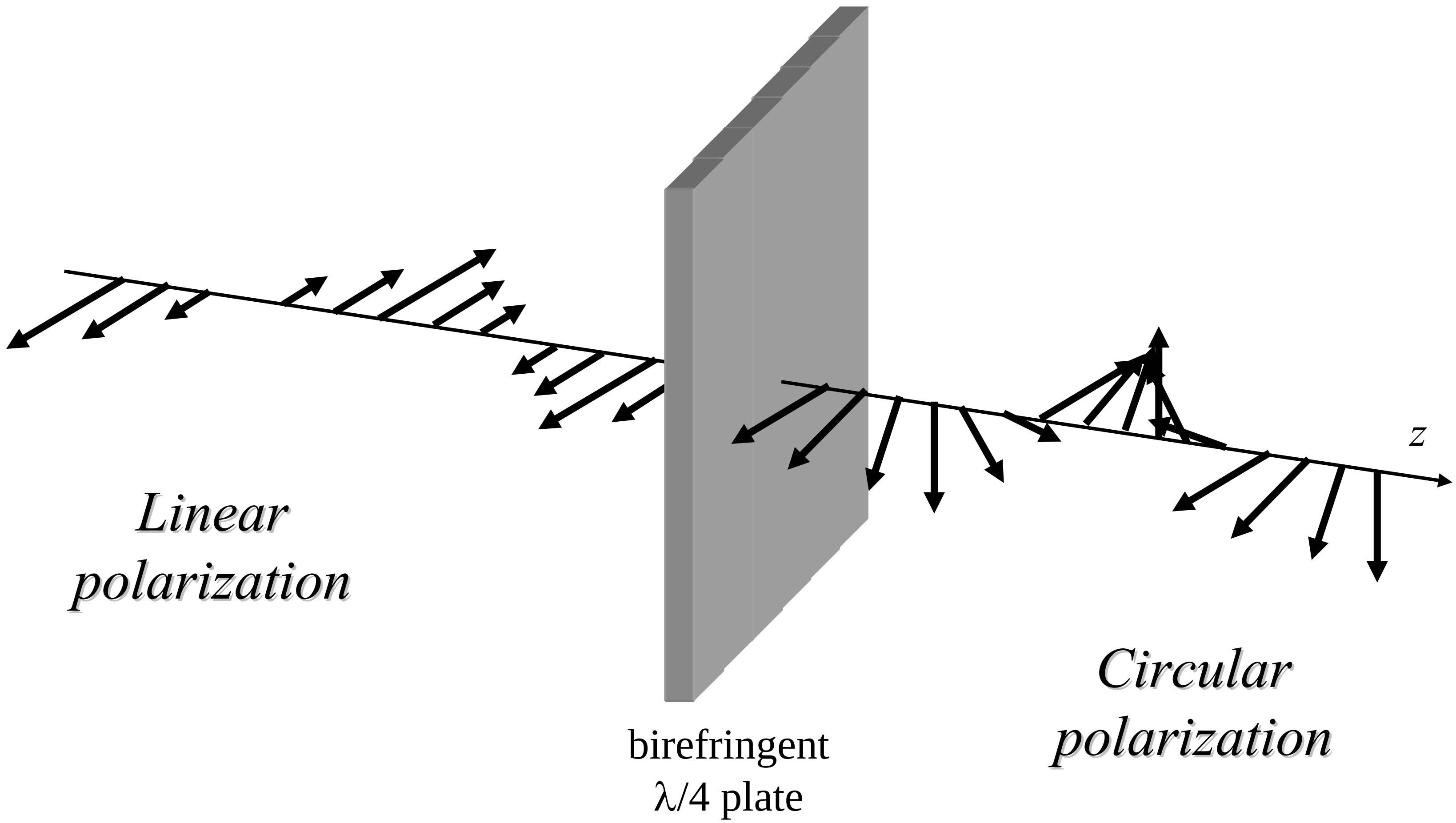
Circular polarization: linear components



Circular polarization (fixed space)

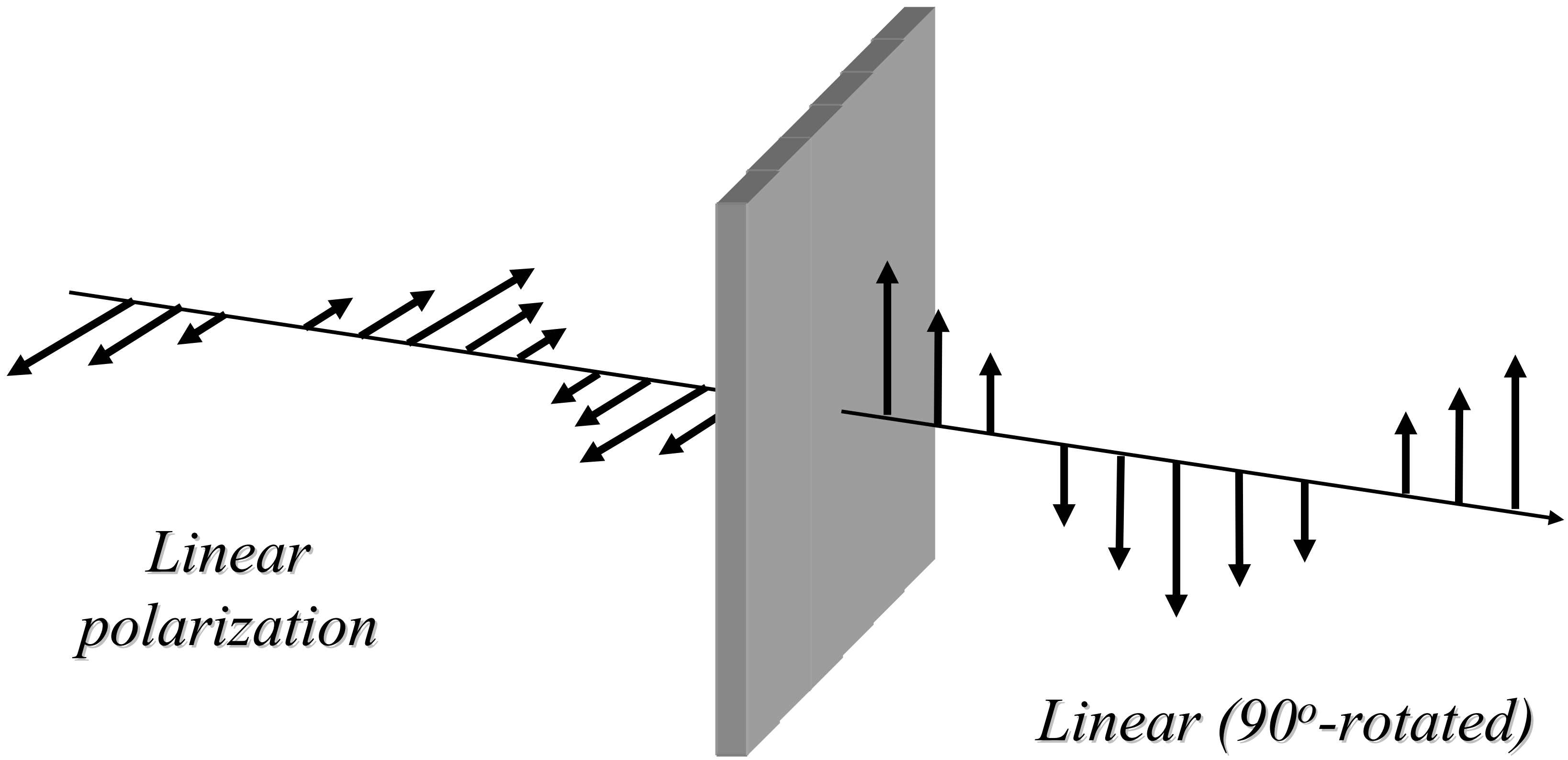


λ/4 plate



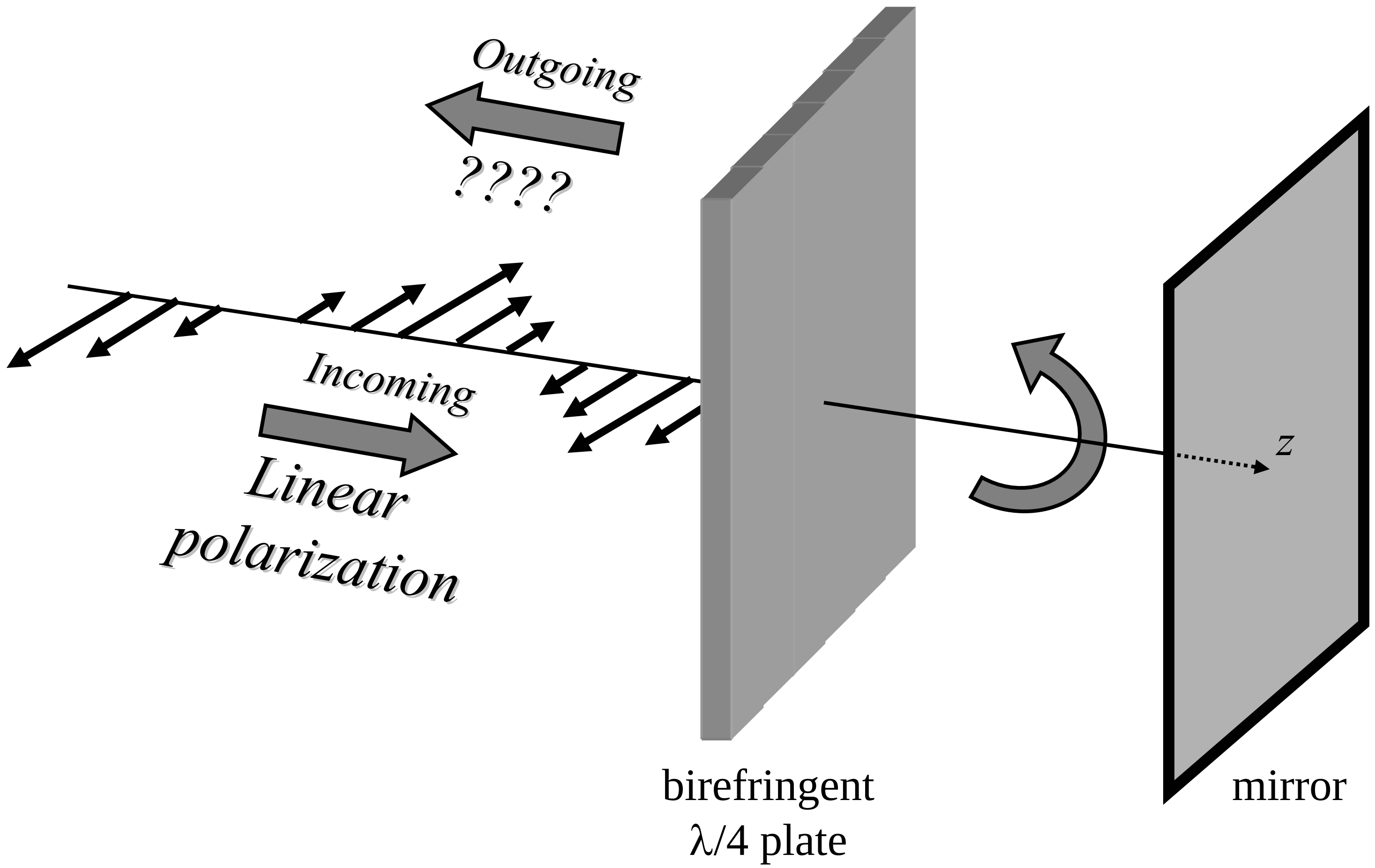
z
Linear polarization
birefringent λ/4 plate
Circular polarization

λ/2 plate

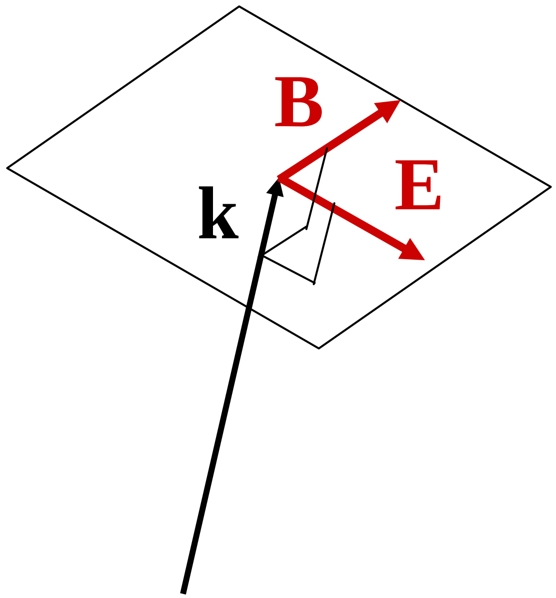


Linear polarization
z
birefringent λ/2 plate
Linear (90°-rotated) polarization

Think about that



Relationship between E and B



$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\quad\text{where}\quad\mathbf{E}=\hat{\mathbf{x}}E_0\,\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$\Rightarrow\nabla\times\equiv i\mathbf{k}\times\quad\text{and}\quad\frac{\partial}{\partial t}\equiv-i\omega$$

$$\Rightarrow\mathbf{B}=\frac{1}{\omega}\mathbf{k}\times\mathbf{E}$$

Vectors **k**, **E**, **B** form a right-handed triad.

Note: free space or isotropic media only