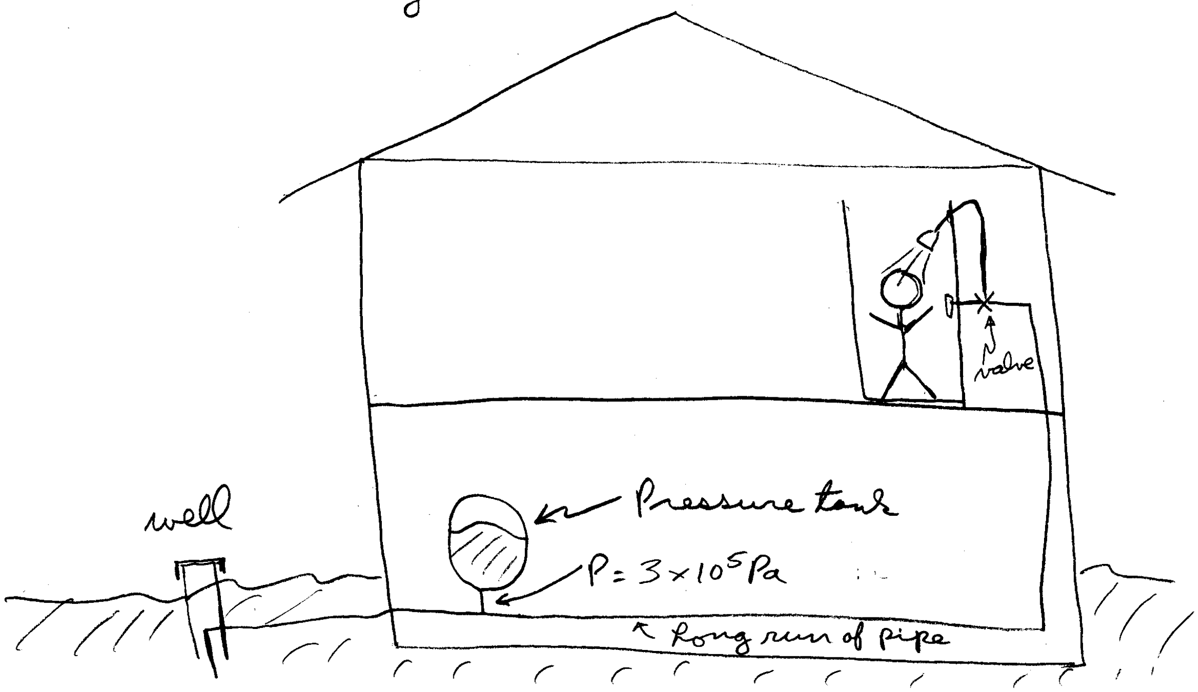
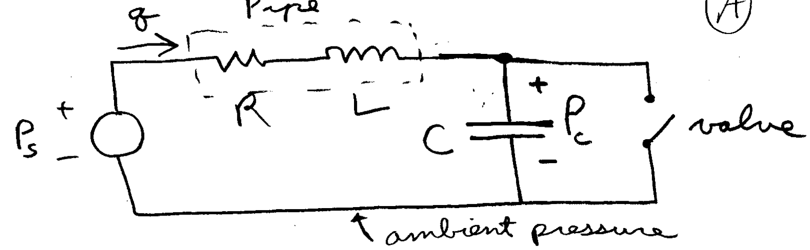
We now consider a fluidic second-order example, which gives some insight into the phenomenon of water-hammer. In your home you may sometimes have noticed a brief banging sound in the walls at the moment when an open faucet ~~or~~ valve is closed suddenly. This water-hammer effect is due to the sudden deceleration of water moving in the pipes. The situation can be represented schematically as shown below



The pressure tank is a fluid accumulator which can be modeled as a fluidic capacitance sufficiently large, in concert with the well pump and pressure controller, to be modeled as a pressure source with a pressure of about 3×10^5 Pa (6895 Pa $= 1$ psi).
Note: Pressure in this problem is measured with respect to atmosphere.

A long run of pipe carries the water upstairs to a shower, where flow is controlled by a valve. We will model the pipe as having a fluidic resistance R and a fluidic inertance L. The valve will be modeled as an on/off switch. A schematic of this connection might appear as follows:

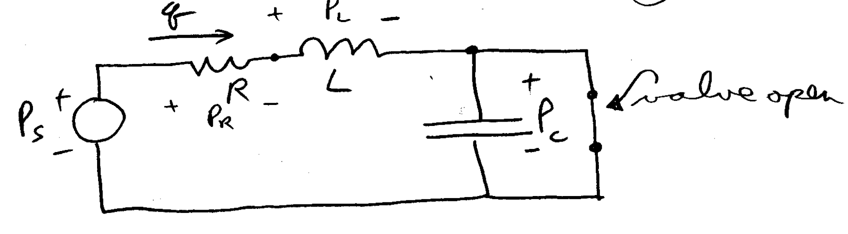
(A)



The fluidic capacitance shown represents the compliance of the pipes and valve, as well as the compressibility of the water plus any entrained air. The flow through the pipe is $q(t)$ [m^3/sec].

We will assume the valve is initially allowing full flow, and thus in this condition we have

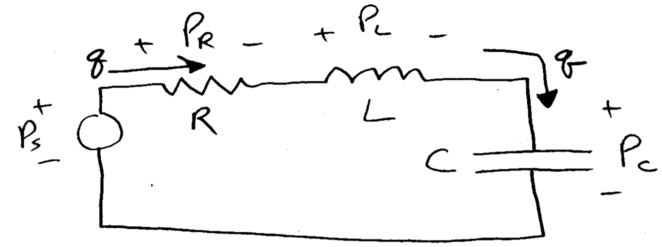
(B)



Thus $q(t<0) = P_s/R$

$P_c(t<0) = 0$

at $t=0$, the valve opens, and the system looks like



The initial conditions for this system are $q(0) = P_s/R$ and $P_c(0) = 0$. Note also that with the valve closed, the flow q is common to all three elements, and thus

$$P_R = qR$$

$$P_L = L\frac{dq}{dt}$$

$$q = C\frac{dP_c}{dt} \Rightarrow P_c = \frac{1}{C}\int q\,dt$$

The connection also imposes $P_c + P_L + P_R = P_s$, and thus

$$qR + L\frac{dq}{dt} + P_c = P_s$$

Substituting from above gives

$$RC\frac{dP_c}{dt} + LC\frac{d^2P_c}{dt^2} + P_c = P_s$$

If we define $P = P_c - P_s$, and $P_s = \text{const}; \frac{dP_s}{dt} = 0$, this can be rewritten as a homogeneous equation:

$$LC \frac{d^2P}{dt^2} + RC \frac{dP}{dt} + P = 0$$

Comparing with the standard second-order form, we see that:

$$LC = \frac{1}{\omega_n^2} \Rightarrow \omega_n = \frac{1}{\sqrt{LC}}$$

$$RC = \frac{2\zeta}{\omega_n} \Rightarrow \zeta = \frac{1}{2} RC\omega_n = \frac{R}{2}\sqrt{\frac{C}{L}}$$

We also need the associated initial conditions. The pressure is

$$\boxed{P(0) = P_c(0) - P_s = -P_s}$$

We also need $\frac{dP}{dt}(0)$. This comes from $q(0)$ as follows. We know ~~that~~

$$q = C\frac{dP_c}{dt} \quad \text{and thus} \quad \frac{dP_c}{dt} = \frac{dP}{dt} = \frac{1}{C}q(0) = \frac{P_s}{RC}$$

$$\Rightarrow \boxed{\frac{dP}{dt}(0) = \frac{P_s}{RC}}$$

Assuming distinct roots, the solution is

$$P(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

where

$$s_1 = \frac{-R}{2L} + \frac{\sqrt{R^2C^2 - 4LC}}{2LC}$$

$$s_2 = \frac{-R}{2L} - \frac{\sqrt{R^2C^2 - 4LC}}{2LC}$$

A reasonable set of numerical parameters might be

$$R = 1.5 \times 10^{9} \ [\text{Pa sec}/m^3]$$

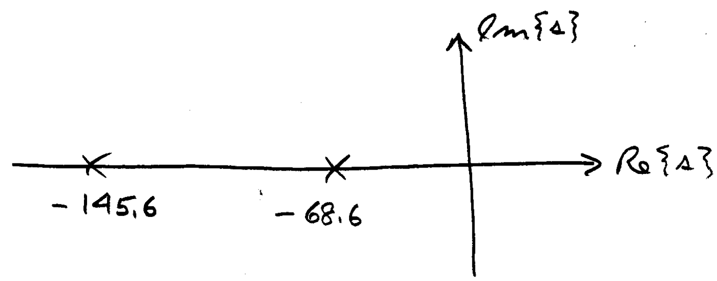
$$L = 7 \times 10^{6} \ [\text{Pa sec}^2/m^3]$$

$$C = \frac{1}{7} \times 10^{-10} \ [m^3/\text{Pa}]$$

and thus $\omega_n = 100$ rad/sec, and $\zeta = 1.07$, which corresponds to an overdamped system. The roots are thus located as:

$$s_1 = -107.1 + 38.5 = -68.6$$

$$s_2 = -107.1 - 38.5 = -145.6$$



The initial conditions are

$$P(0) = -P_s = -3 \times 10^5 = C_1 + C_2 \quad [\mathrm{Pa}]$$

$$\frac{dP}{dt}(0) = \frac{P_s}{RC} = \frac{3 \times 10^5}{1.5 \times 10^9 \cdot \frac{1}{7} \times 10^{-10}} = 1.4 \times 10^7 \ [\mathrm{Pa/sec}]$$

$$= C_1 s_1 + C_2 s_2$$

From the first relationship:

$$C_2 = P(0) - C_1$$

Plugging into the second gives

$$\dot{P}(0) = C_1 s_1 + (P(0) - C_1) s_2$$

and thus

$$\dot{P}(0) - P(0) s_2 = C_1 (s_1 - s_2)$$

and so

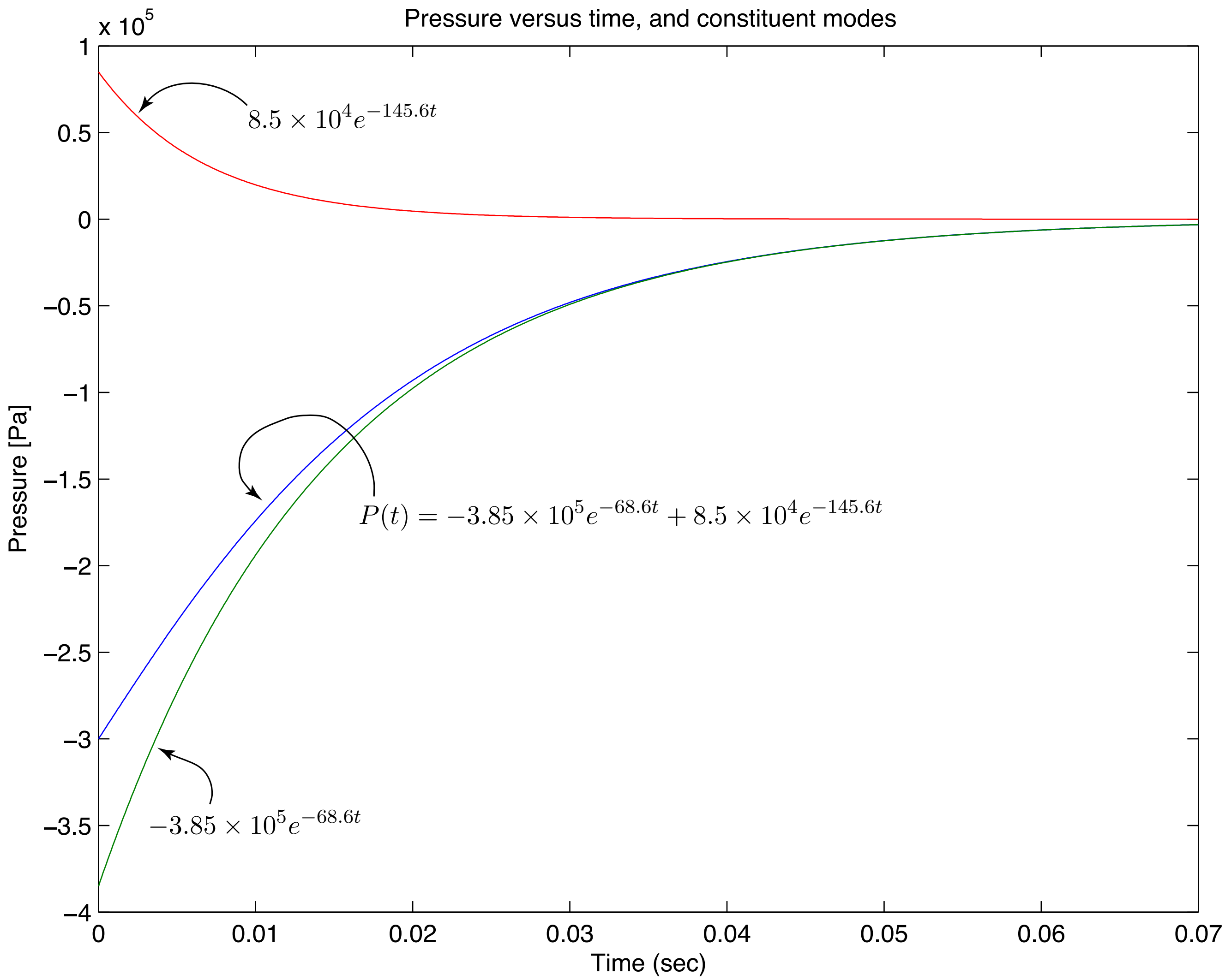
$$C_1 = \frac{\dot{P}(0) - P(0) s_2}{s_1 - s_2} = -3.85 \times 10^5 \ [\mathrm{Pa}]$$

$$C_2 = P(0) - C_1 = 8.5 \times 10^4 \ [\mathrm{Pa}]$$

And thus the pressure as a function of time is

$$P(t) = -3.85 \times 10^5 e^{-68.6t} + 8.5 \times 10^4 e^{-145.6t}$$

A plot of this time function and the two constituent modes is shown on the next page.


Pressure versus time, and constituent modes
x 10^5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
−3.5
−4
Pressure [Pa]
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
Time (sec)
$8.5 \times 10^4 e^{-145.6t}$
$P(t) = -3.85 \times 10^5 e^{-68.6t} + 8.5 \times 10^4 e^{-145.6t}$
$-3.85 \times 10^5 e^{-68.6t}$

Note that pressure $P(t)$ varies rapidly from an initial value of $P(0) = -3 \times 10^5$ [Pa] to exponentially approach a final value of $P(\infty) = 0$. Since we have defined $P(t) = P_c(t) - P_s$, this means that the pressure at the valve varies rapidly from an initial value of $P_c(0) = 0$ to a final value of $P_c(\infty) = P_s = 3 \times 10^5$ Pa. This sudden variation in pressure exerts a near step in force on the end of the pipe of $F = P_c \cdot A$, where A is the pipe area. If the pipe ID is 2×10^{-2} m, then $A = \frac{\pi D^2}{4} \cong 3 \times 10^{-4}$ m, and thus $F = 3 \times 10^5 \cdot 3 \times 10^{-4} = 90$ N. This significant force is applied to the valve body which is supported by the wall framing members. The step load on the wall results in the characteristic banging sound. (Similar forces also occur where the pipe makes turns in its path through the walls, and thus you hear banging throughout the plumbing system when you suddenly shut off the shower.) Be kind to your plumbing and turn the valve off a bit slowly!