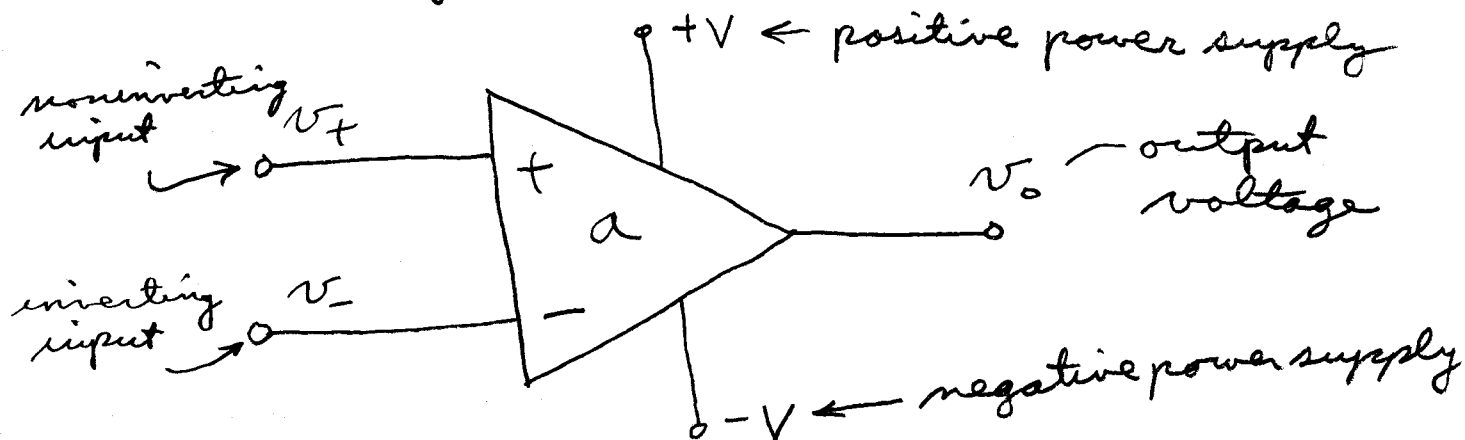


Operational Amplifiers

①

Operational amplifiers are highly useful circuit building blocks. This is so because their behavior is nearly ideal over a wide range of operating conditions, and they can be used without worrying too much about the internal circuitry of the device.

The symbol for an op amp is



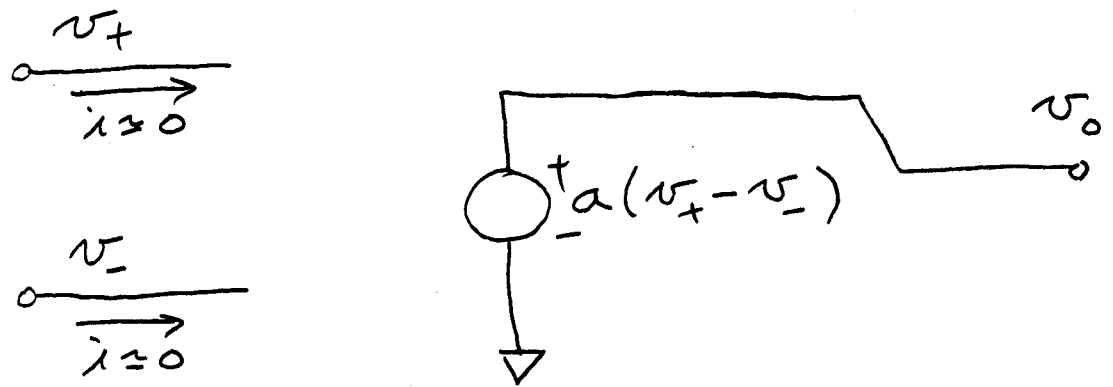
all the voltages above are defined with respect to a common or ground connection not shown in the figure.

Op amps are an active device in the sense that the output voltage is created by using the power supplies $+V$ and $-V$ to drive the output V_o to some desired level.

Meanwhile, the inverting and noninverting input terminals draw almost no input

(2)

current (on the order of nA or less) and thus require almost no input power. A circuit equivalent for the op amp is



Here the output voltage is modeled as generated by a dependent voltage source. That is,

$$v_o = a(v_+ - v_-) \quad (1)$$

That is, the output voltage is given by the differential input voltage $v_+ - v_-$ multiplied by a scale factor a . The internal circuitry which accomplishes this is fascinating, but we will not go into that in these notes.

(3)

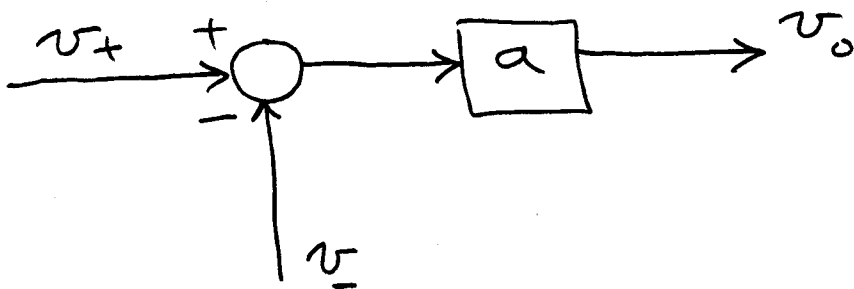
From this equivalent circuitry, we will consider 3 models:

- i) a is infinite
- ii) a is finite, but just a constant (no dynamics)
- iii) $a \Rightarrow a(s)$, a transfer function (dynamics internal to op amp)

The choice of which model to use depends upon a particular application, but we should always be clear which model is being used.

Block diagram:

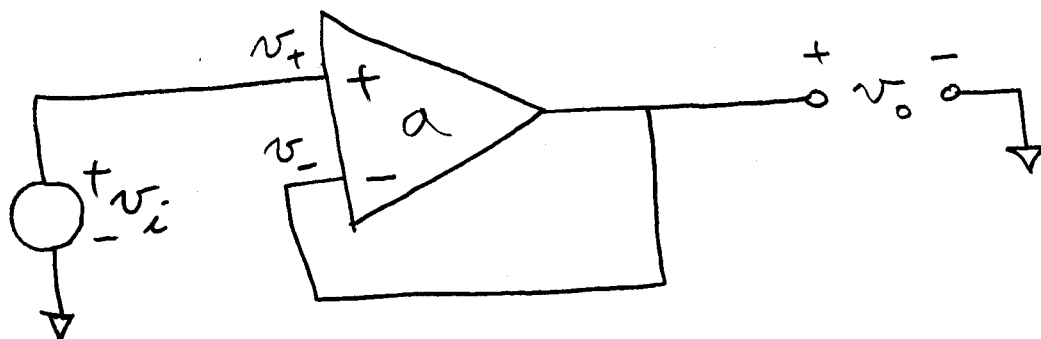
The relationship (1) can also be modeled in the form of a block diagram as



Unity-gain buffer

4

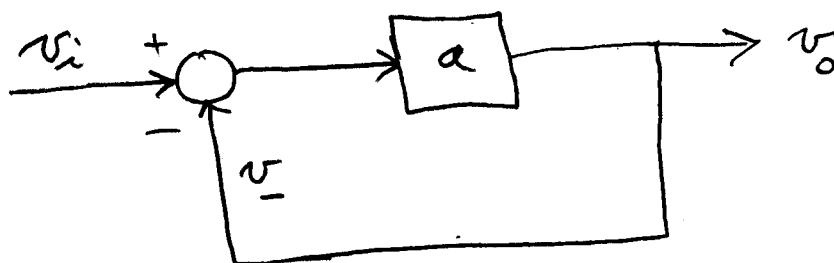
A common configuration is the unity gain buffer. First, consider this with the finite-gain model i).



Here, we have :

$$v_+ = v_i$$
$$v_- = v_o$$
$$v_o = a(v_+ - v_-)$$

Or, in block diagram form:



via Black's formula, we have

$$v_o = \frac{a}{1+a} v_i$$

If $a = 10^5$, a typical value :

$$\frac{v_o}{v_i} = \frac{10^5}{1+10^5} = 0.99999 \approx 1$$

(5)

The larger a becomes, the closer the closed-loop gain approaches a value of 1. For example, if $a = 10^6$,

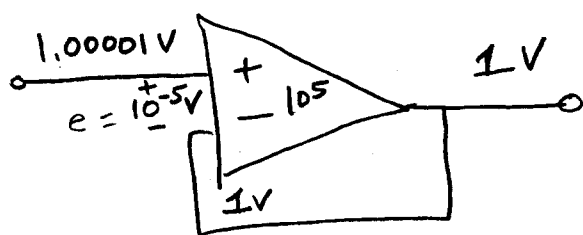
$$\frac{v_o}{v_i} = \frac{10^6}{1 + 10^6} = 0.999999$$

and finally, using model ii)

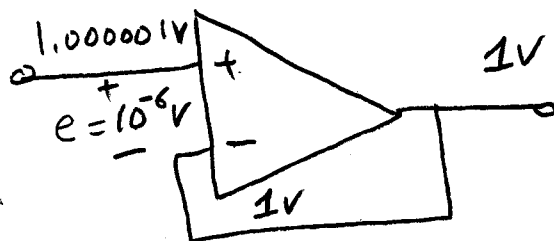
$$\lim_{a \rightarrow \infty} \frac{v_o}{v_i} = 1$$

So high amplifier gain is good for closed loop accuracy.

In the case ii) where $a \rightarrow \infty$, we can solve the circuit fairly directly for the closed-loop characteristics. The key is to recognize that for finite v_o , as $a \rightarrow \infty$ ($v_+ - v_- \rightarrow 0$). That is, in the limit of large gain, the differential input voltage must approach zero, as shown below.



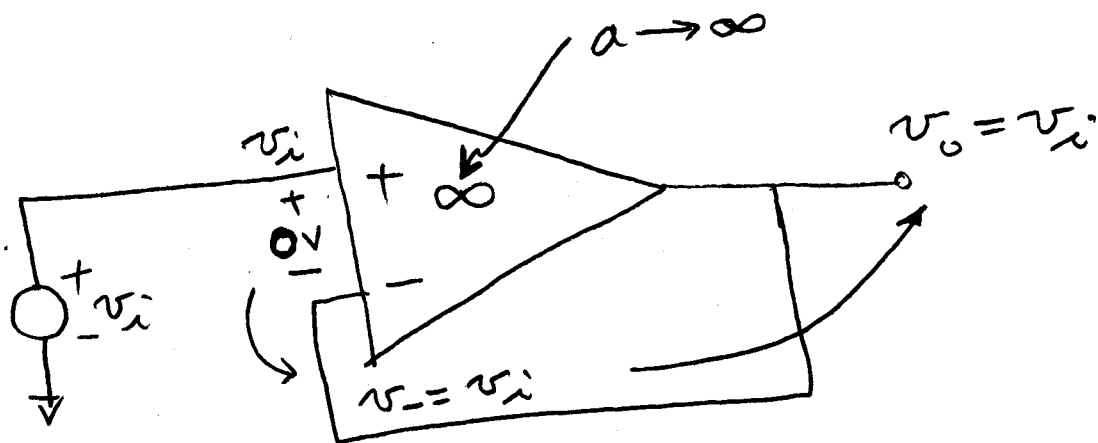
$$a = 10^5 \Rightarrow e = 10\mu\text{V}$$



$$a = 10^6 \Rightarrow e = 1\mu\text{V}$$

(6)

With this insight, the circuit can be solved directly as follows:



That is, since $v_+ - v_- = 0 \Rightarrow v_- = v_+ = v_i$, and thus $v_o = v_- = v_i$. This is called the zero differential input method.

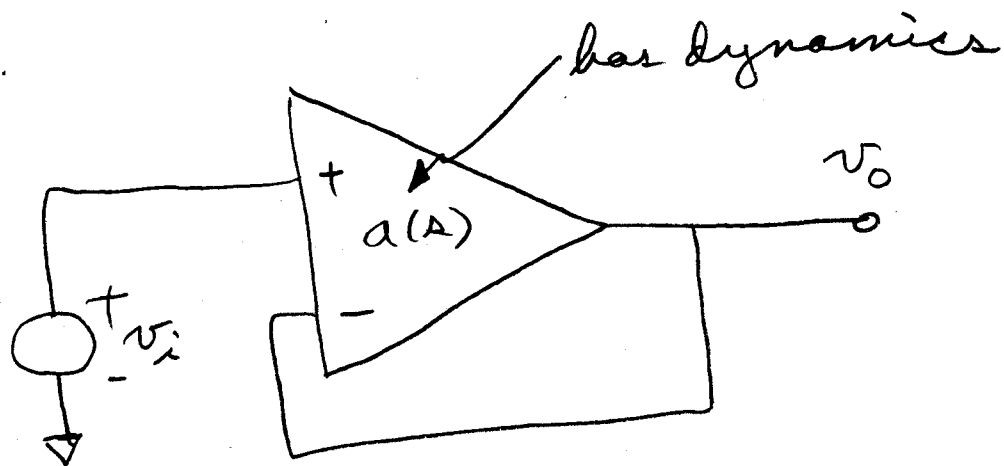
Finally, we look at model iii) where $a = a(s)$, and the amplifier has internal dynamics. Ultimately real devices must have some sort of dynamics associated with them, since all real objects have some finite time response characteristics.

Op amps have remarkably ideal dynamics. To a very good approximation we can model most op amps with the transfer function $a(s) = g/s$. That is,

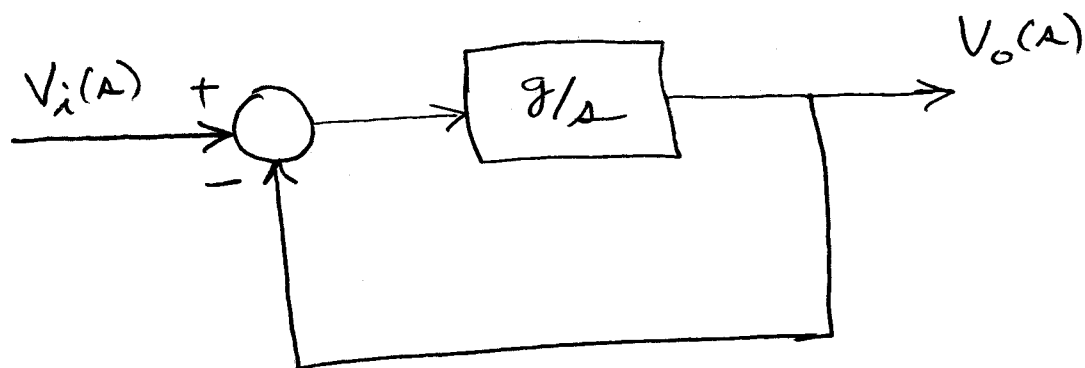
$$V_o(s) = g/s (V_+(s) - V_-(s))$$

(7)

So, most of amps are intentionally designed to act as an integrator of differential input voltage. In the case of the unity gain buffer:



The block diagram is



via Black's formula:

$$\frac{V_o(s)}{V_i(s)} = \frac{g/s}{1 + g/s} = \frac{1}{\frac{s}{g} + 1} = \frac{1}{\tau s + 1}$$

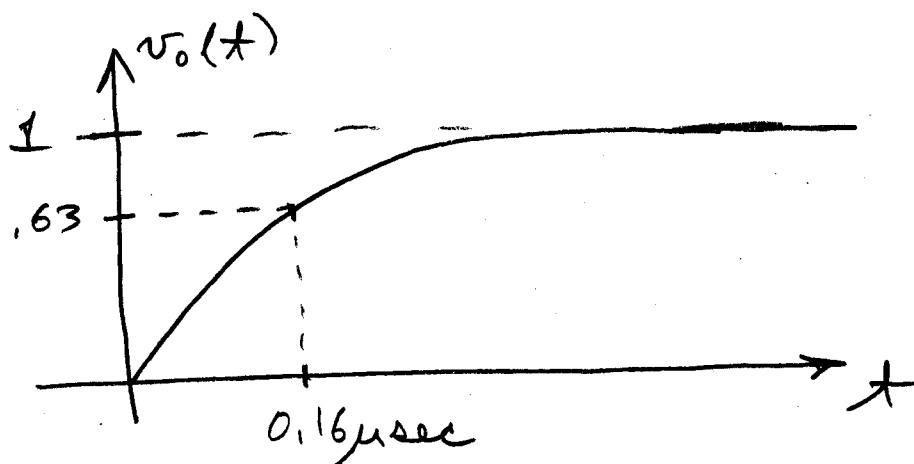
$$\tau = \frac{1}{g}$$

(8)

In a typical moderate-performance op amp $g = 2\pi \times 10^6$ (for example an LM741) and thus

$$\tau = \frac{1}{2\pi \times 10^6} \approx 1.6 \times 10^{-7} \text{ sec}$$

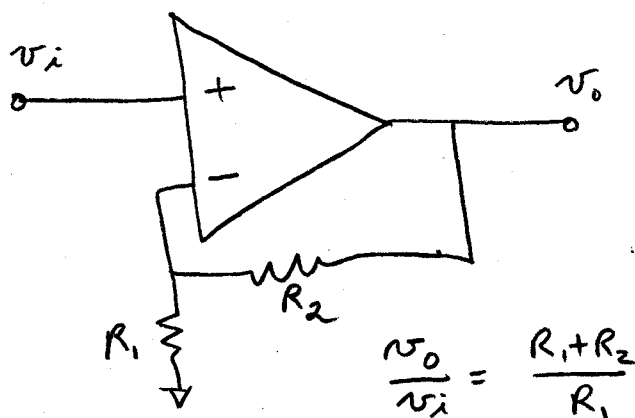
Thus the step response will look like



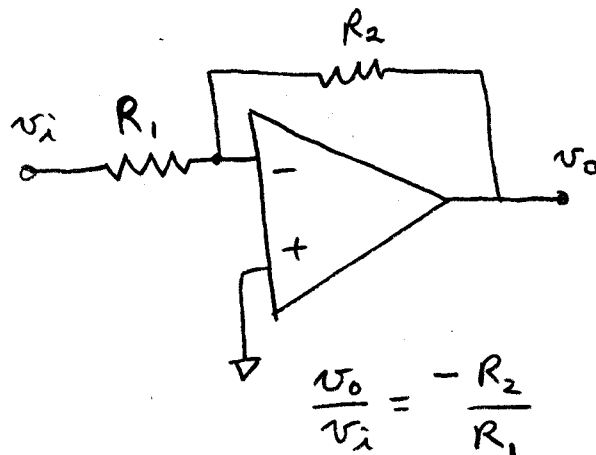
This is pretty fast relative to mechanical systems, but not all that fast for an electrical system.

Two other configurations of interest are

Noninverting



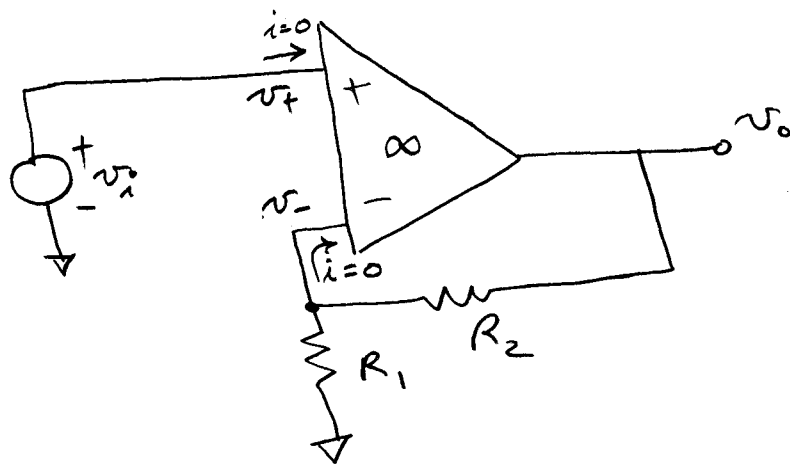
Inverting



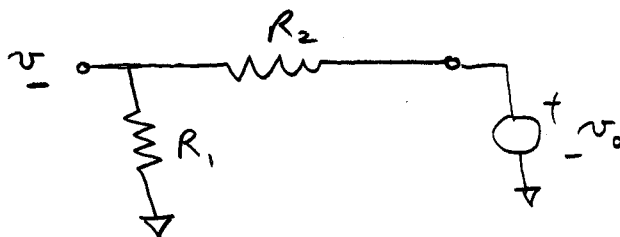
Note that feedback in both cases is taken to the inverting terminal of the op amp. This is because negative feedback is in general stabilizing whereas positive feedback is in general destabilizing. Think of driving a car. If you drift to the right on the road you follow a negative feedback strategy by turning the wheel (your actuator input) to the left. The alternative positive feedback strategy would have you turn the wheel more to the right the further you drift to the right. This clearly diverges rapidly into a trip into the weeds! So we will generally prefer negative feedback systems. Positive feedback has its uses which are quite important. For example, the latching mechanism of a suitcase is a positive feedback device as you are closing it. This results in the latch staying closed despite disturbance loads.

We now turn to deriving the input/output relationships for the inverting and noninverting amplifiers. We first use model ii), the infinite gain assumption, the model i) with finite gain, and finally model iii) with dynamics.

Noninverting amplifier:



Recall that the inputs of an op amp draw essentially zero current. Thus, we can compute v_- as though the op amp isn't there. The circuit that goes with this is



and thus by voltage divider we have

$$v_- = v_o \frac{R_1}{R_1 + R_2}$$

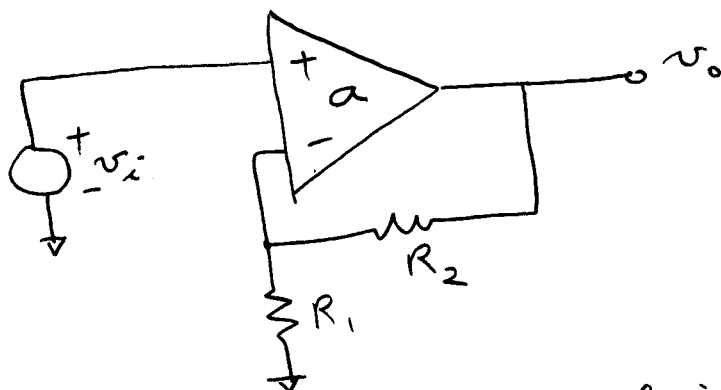
Now, since $a \rightarrow \infty$, it must hold that $v_+ = v_-$, and thus $v_- = v_i$. Thereby we find

$$v_o = v_i \frac{R_1 + R_2}{R_1}$$

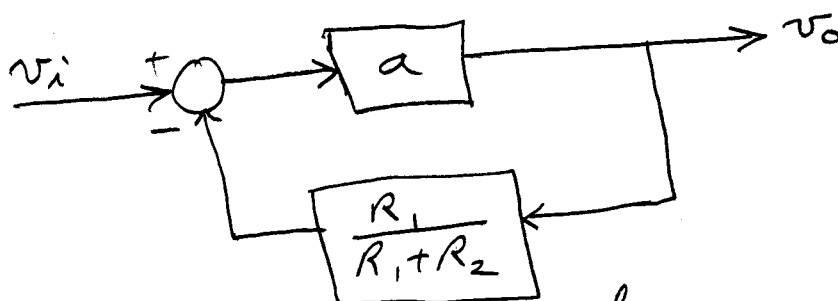
One way to think of this is that the feedback loop drives v_o such that $v_- = v_+$, and thereby via the voltage divider, the more we divide v_o down into v_- , the larger the gain $v_o/v_i = (R_1 + R_2)/R_1$.

Now, we switch to a finite-gain model ii) for the noninverting amplifier

(11)



The voltage divider result is the same as before $v_- = v_o \frac{R_1}{R_1 + R_2}$. Also, $v_+ = v_i$; and $v_o = a(v_+ - v_-)$. The block diagram which goes with this is



Via Block's formula, we have

$$\frac{v_o}{v_i} = \frac{a}{1 + \frac{a R_1}{R_1 + R_2}} = \frac{a}{\frac{R_1}{R_1 + R_2} \left(\frac{R_1 + R_2}{R_1} + a \right)}$$

$$= \underbrace{\frac{R_1 + R_2}{R_1}}_{\text{Ideal C.L. gain}} \cdot \underbrace{\left(\frac{a}{\frac{R_1 + R_2}{R_1} + a} \right)}_{\text{nonideal gain term } < 1} \quad (2)$$

Ideal
C.L. gain

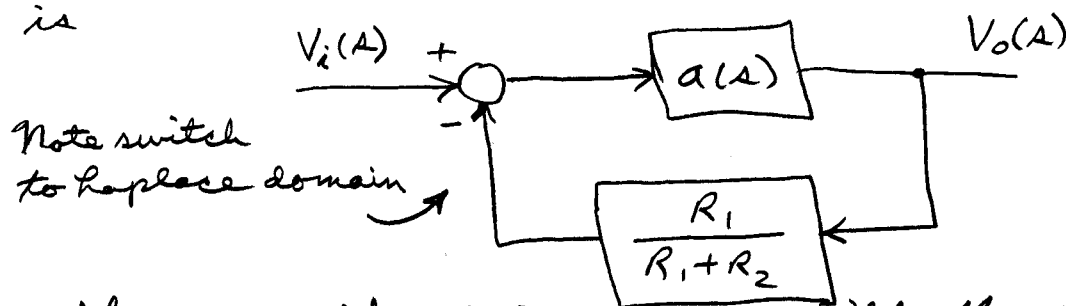
nonideal gain term < 1
(approaches 1 as $a \rightarrow \infty$)

Thus we see that the input output gain can be expressed as the product of an ideal gain term $\frac{R_1 + R_2}{R_1}$ and an attenuating nonideal gain term. This second term approaches unity as $a \rightarrow \infty$. So we see that for closed-loop gain accuracy we need $a \gg \frac{R_1 + R_2}{R_1}$.

Said another way we need $\frac{a R_1}{R_1 + R_2} \gg 1$.

That is, the magnitude of the loop transmission must be much larger than 1 if we want good closed-loop gain accuracy. The loop transmission is the key factor for feedback loop performance.

We now turn to the model iii) with dynamics, where $a \rightarrow a(s)$. The block diagram is



Thus, result (2) is valid with the simple substitution of $a(s)$ for a . That is,

$$\frac{V_o(s)}{V_i(s)} = \left(\frac{R_1 + R_2}{R_1} \right) \left(\frac{a(s)}{\frac{R_1 + R_2}{R_1} + a(s)} \right)$$

If we let $a(s) = g/s$, a reasonable model:

$$\frac{V_o(s)}{V_i(s)} = \left(\frac{R_1 + R_2}{R_1} \right) \left(\frac{1}{\frac{R_1 + R_2}{g R_1} s + 1} \right)$$

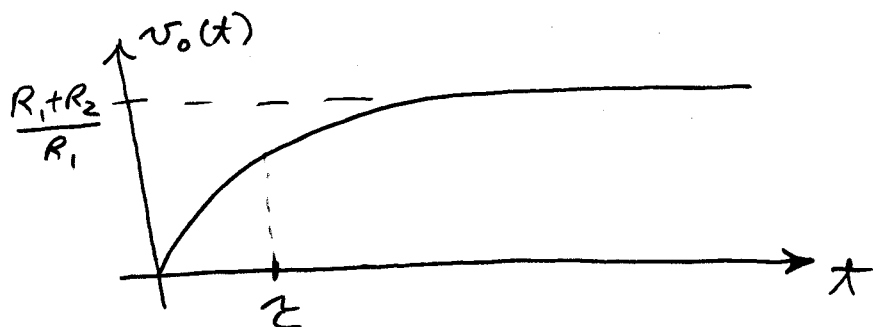
If we define $\tau = \frac{R_1 + R_2}{R_1 g}$, then this becomes

$$\frac{V_o(s)}{V_i(s)} = \left(\frac{R_1 + R_2}{R_1} \right) \left(\frac{1}{\tau s + 1} \right)$$

the product of an ideal gain factor with a first-order transfer function.

This transfer function has a unit step response given by (you should verify this)

$$v_o(t) = \frac{R_1 + R_2}{R_1} (1 - e^{-t/\tau}) ; t > 0$$



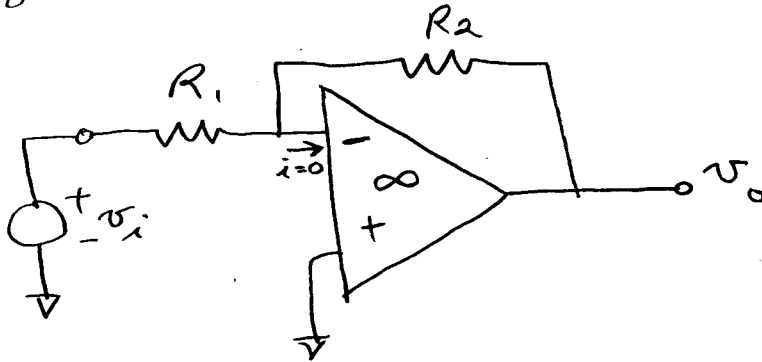
Note that the final value and time constant are both linearly dependent on the gain term $\frac{R_1 + R_2}{R_1}$. That is, as you ask for larger closed-loop DC gain, the time constant becomes larger and the system slows down. Assume $g = 10^7$. Then

DC gain $\rightarrow \frac{R_1 + R_2}{R_1}$	τ (μsec) $\leftarrow$ closed-loop time constant
2	0.2
10	1
100	10

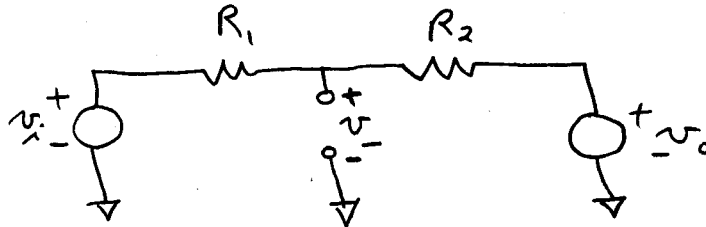
Lesson: Feedback modifies dynamics.

Next we look at the inverting amplifier configuration with model i), $a \rightarrow \infty$.

(14)



With v_+ connected to ground, $v_+ = 0$, and with $a \rightarrow \infty$, $v_- = v_+ = 0$. Since no current flows into the op amp, the resistive circuit can be analysed as



Via superposition and the voltage divider relationship, we find

$$v_- = v_i \frac{R_2}{R_1 + R_2} + v_o \frac{R_1}{R_1 + R_2} \quad (3)$$

With $v_- = 0$, this yields

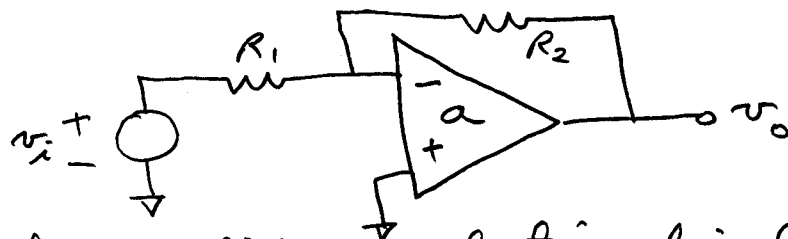
$$v_i \frac{R_2}{R_1 + R_2} = -v_o \frac{R_1}{R_1 + R_2}$$

and thus

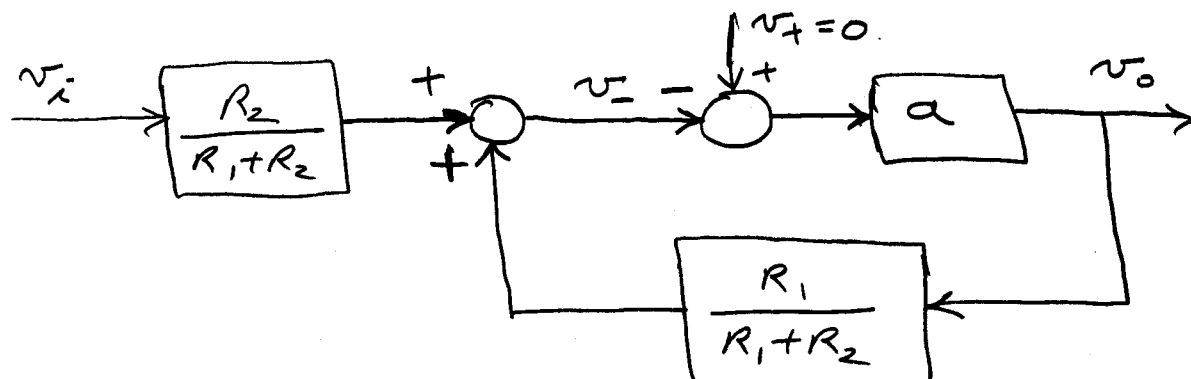
$$\boxed{\frac{v_o}{v_i} = -\frac{R_2}{R_1}}$$

the ideal gain relationship for an inverting amplifier.

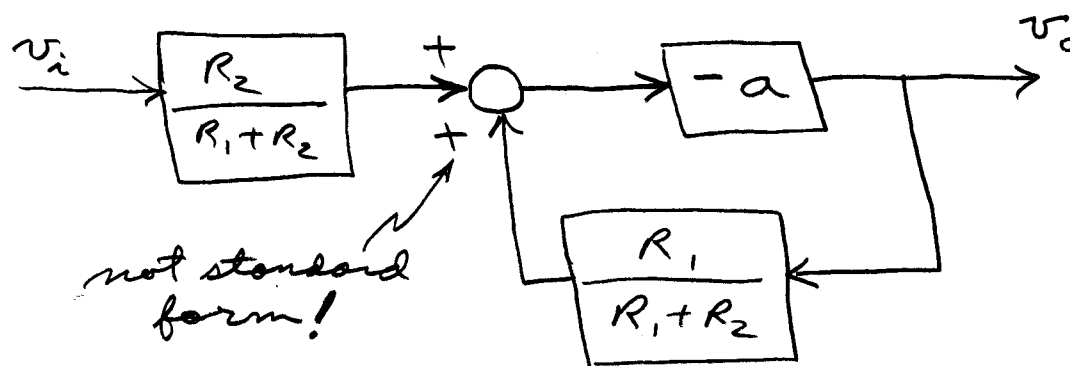
Next we turn to the finite gain model ii).



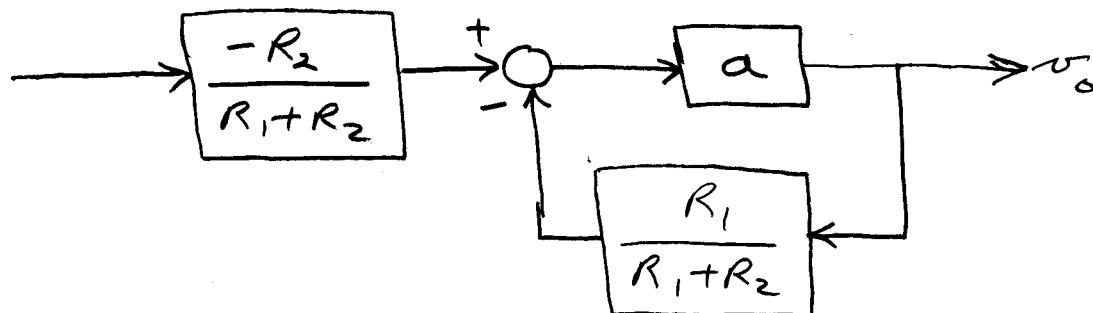
The voltage-divider relationship (3) still holds, and thus the associated block diagram is



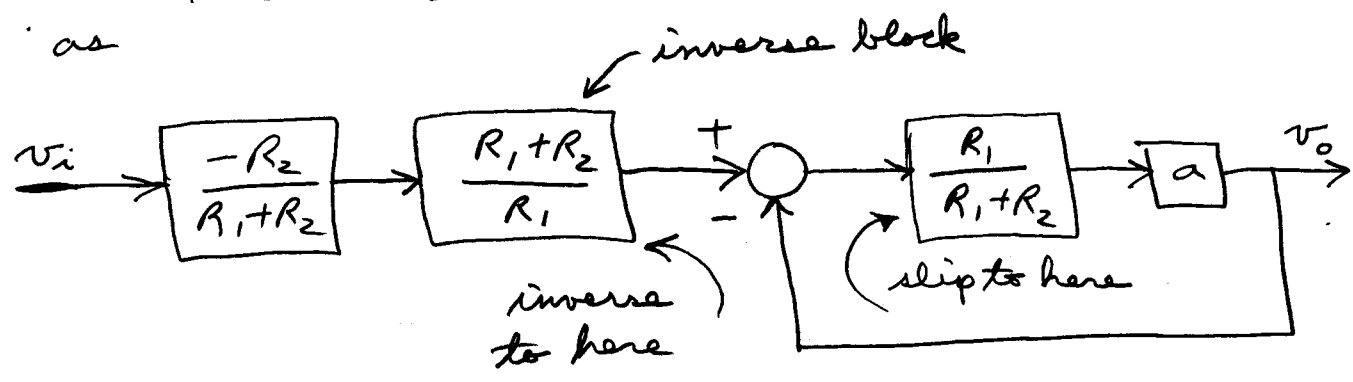
This simplifies to ($v_+ = 0$):



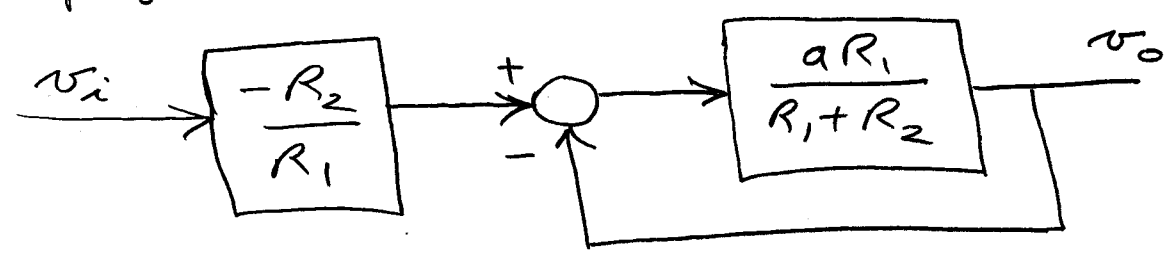
If we pull the minus sign through from a , this can be transformed to standard form as



At this point we could apply Block's formula and solve for v_o/v_i ; but it's instructive to see how some of the algebra can be accomplished via block diagram manipulation. Specifically, I'd like to transform the loop to unity-feedback form. We think of sliding the feedback block $R_1/(R_1+R_2)$ along through the summing junction as



Note that in sliding the feedback block into the forward path, we have to slide its inverse into the input path. This diagram then simplifies as



and thus

$$\frac{v_o}{v_i} = \frac{-R_2}{R_1} \cdot \frac{\frac{aR_1}{R_1+R_2}}{1 + \frac{aR_1}{R_1+R_2}}$$

Ideal gain = $\underbrace{-\frac{R_2}{R_1}}_{\text{Ideal gain}} \cdot \underbrace{\frac{a}{\frac{R_1+R_2}{R_1} + a}}_{\text{Nonideal gain factor}}$

Nonideal gain factor $\Rightarrow 1$ as $a \Rightarrow \infty$

As before we have written v_o/v_i as the product of the ideal ($a \rightarrow \infty$) gain times an attenuating gain factor. As before, this nonideal factor approaches unity as $a \rightarrow \infty$. To have gain accuracy, we need $a \gg \frac{R_1 + R_2}{R_1}$, or equivalently $\frac{a R_1}{R_1 + R_2} \gg 1$. This is the same accuracy criterion that we found for the noninverting amplifier, which might at first seem surprising. However, if you look at the block diagrams for these two configurations, you'll see that they have the same loop transmissions $L.T. = \frac{-a R_1}{R_1 + R_2}$, and thus the same accuracy criteria.

Finally, we use model iii) where $a \rightarrow a(s)$, and thus

$$\frac{V_o(s)}{V_i(s)} = \frac{-R_2}{R_1} \cdot \frac{a(s)}{\frac{R_1 + R_2}{R_1} + a(s)}$$

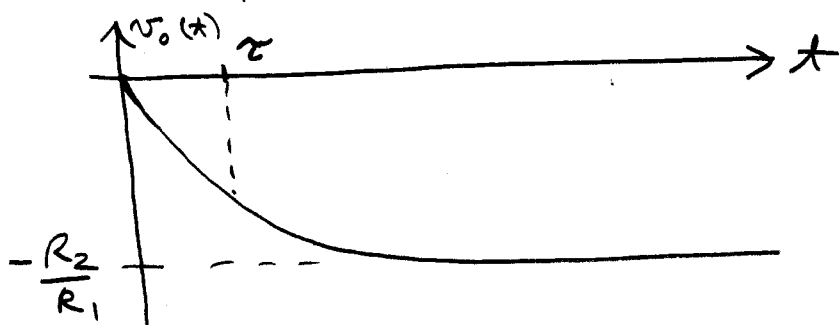
(be sure to verify this if you don't see where it came from)

If we let $a(s) = g/s$, then

$$\frac{V_o(s)}{V_i(s)} = \frac{-R_2}{R_1} \cdot \frac{1}{\tau s + 1} ; \tau = \frac{R_1 + R_2}{R_1 g}$$

This has an associated unit step response of

$$v_o(t) = -\frac{R_2}{R_1} (1 - e^{-t/\tau}) ; t > 0$$



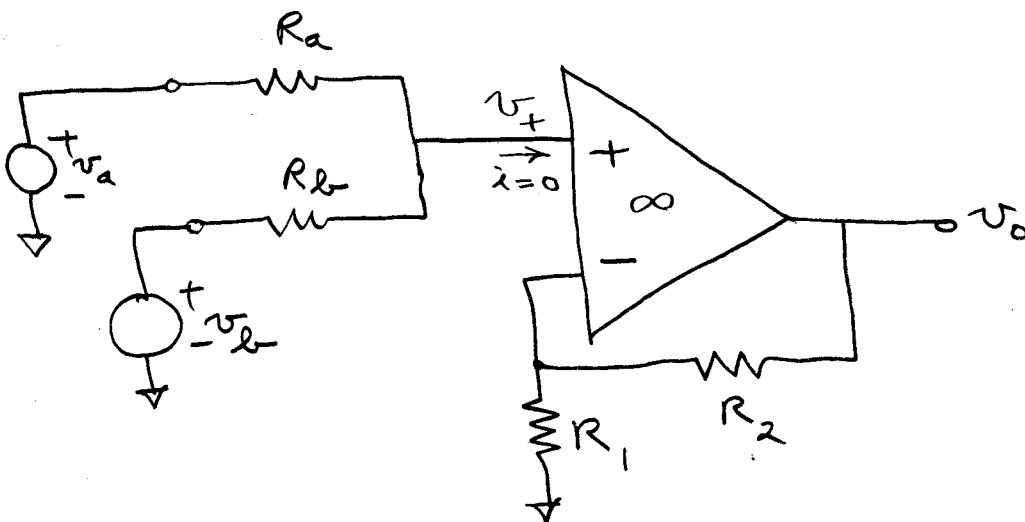
(18)

As before, the closed-loop time constant depends upon the chosen gain. If $g = 10^7$:

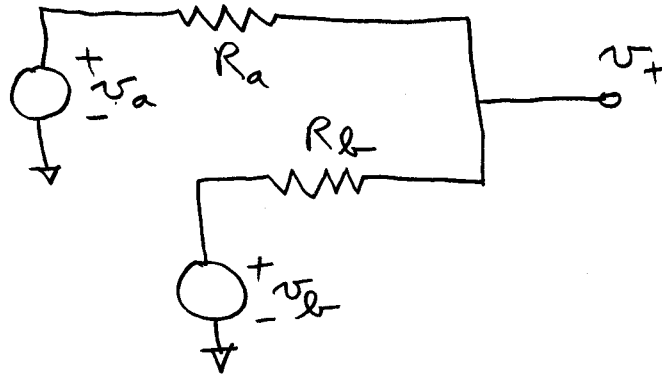
$\frac{R_2}{R_1}$	$\frac{R_1 + R_2}{R_1}$	τ (μsec)
1	2	0.2
9	10	1
99	100	10

Summation amplifiers:

A very useful application for op amps is to add and subtract signals. For example, consider the non-inverting summer shown below:



Since, ^{we assume} no current flows into the op amp, we can solve for v_+ as though the amplifier isn't there. The related circuit is



Via superposition and voltage divider, we find

$$v_+ = v_a \frac{R_b}{R_a + R_b} + v_b \frac{R_a}{R_a + R_b}$$

Now, from v_+ on, it's just a standard noninverting amplifier configuration. That is

$$v_o = v_+ \frac{R_1 + R_2}{R_1}$$

and thus

$$v_o = \frac{R_1 + R_2}{R_1} \left(v_a \frac{R_b}{R_a + R_b} + v_b \frac{R_a}{R_a + R_b} \right)$$

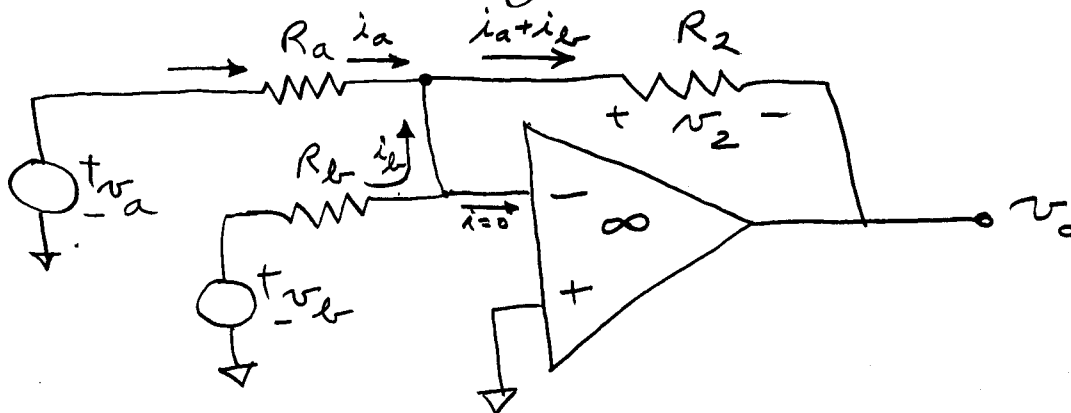
which gives an output equal to the weighted sum of the two inputs. For example, if we choose $R_a = 1k\Omega$, $R_b = 4k\Omega$, $R_1 = 1k\Omega$, and $R_2 = 9k\Omega$, then

$$v_o = 10 \left(v_a \cdot \frac{4}{5} + v_b \cdot \frac{1}{5} \right) = 8v_a + 2v_b$$

This idea readily extends to N inputs by having N resistors at the input.

Inverting summer:

The inverting summer configuration is



Because we assume $a \rightarrow \infty$, and since $v_+ = 0$, then $v_- = v_+ = 0$. This means we can calculate the input resistor currents as

$$i_a = v_a / R_a$$

$$i_b = v_b / R_b$$

Since no current flows into the amplifier, these two currents must sum and flow through R_2 as shown. Thus $v_2 = (i_a + i_b) R_2$. But since $v_- = 0$, we must have $v_o = -v_2 = -(i_a + i_b) R_2$ and thus

$$v_o = -\left(\frac{v_a}{R_a} + \frac{v_b}{R_b}\right) R_2$$

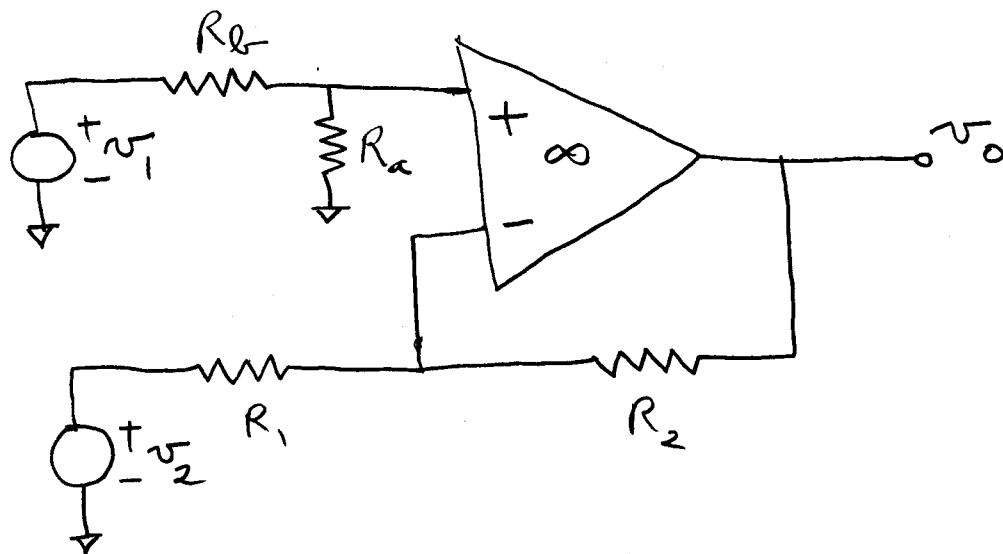
For example, if we choose $R_2 = 9\text{ k}\Omega$, $R_a = 1\text{ k}\Omega$, and $R_b = 3\text{ k}\Omega$, then

$$v_o = -\left(\frac{v_a}{1\text{ k}\Omega} + \frac{v_b}{3\text{ k}\Omega}\right) \cdot 9\text{ k}\Omega = -9v_a - 3v_b$$

The idea extends to N inputs by using N input resistors.

Differential amplifier :

Another important configuration is the differential amplifier

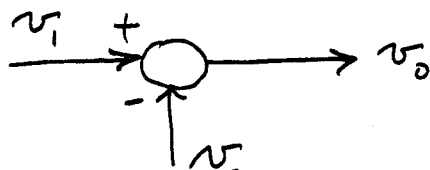


Via superposition you can show that (you should try to derive this)

$$v_0 = -\frac{R_2}{R_1} v_2 + \frac{R_1 + R_2}{R_1} \cdot \frac{R_a}{R_a + R_b} v_1$$

In particular, if $R_1 = R_2 = R_a = R_b$, we have

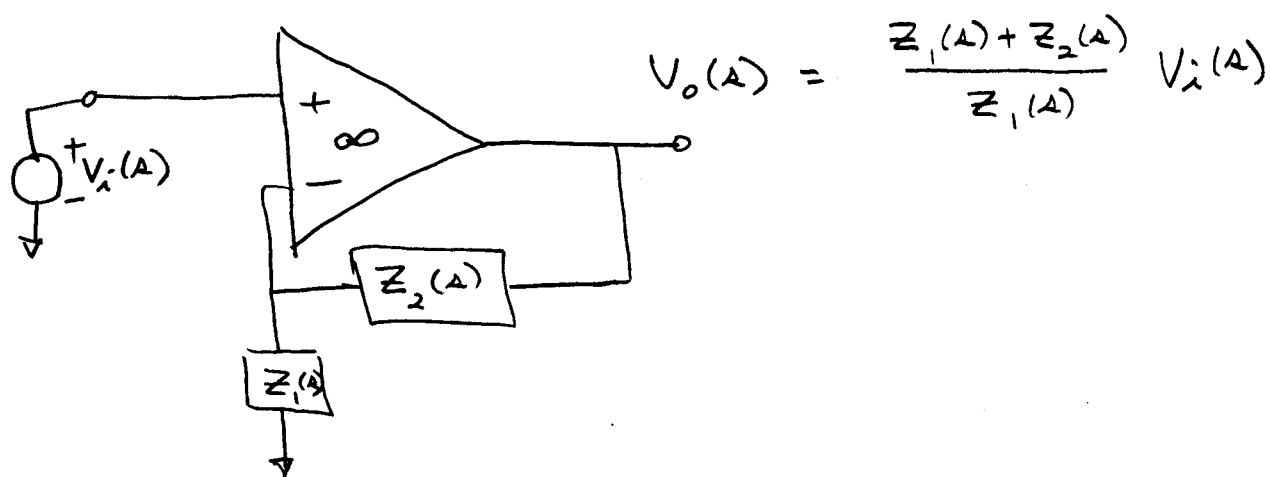
$v_0 = v_1 - v_2$; i.e. the difference of the two inputs. This has a block diagram



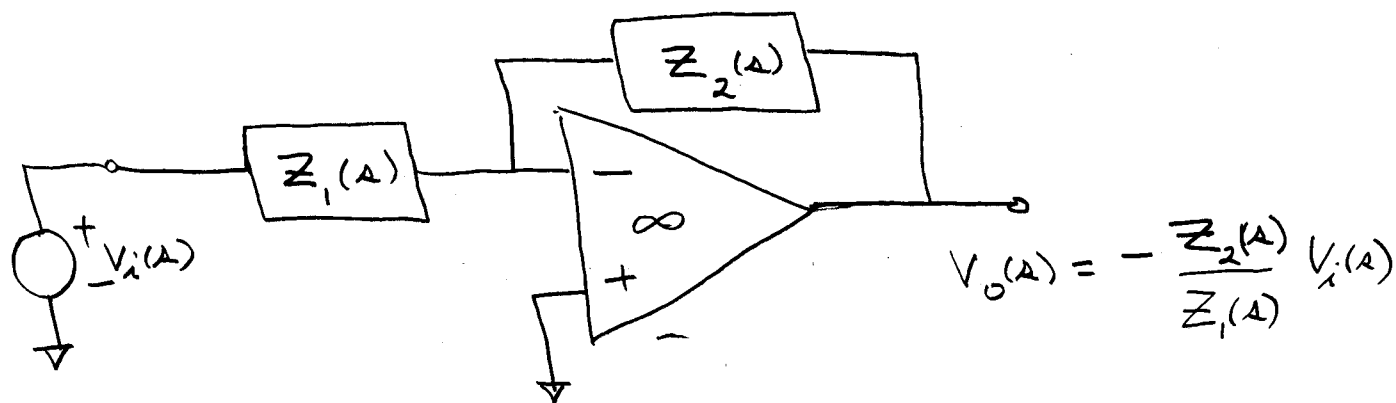
Now you see how to make an electronic summing junction.

Filters:

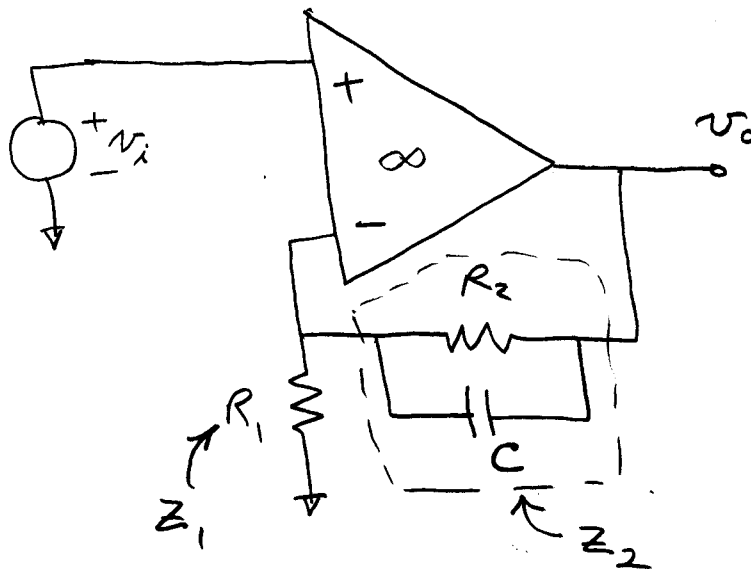
So far all the op amp circuits we've discussed have used resistive feedback. If we replace those resistors with impedances, the gain formulae derived earlier are applicable



and



For example:



$$Z_1 = R_1$$

$$Z_2 = R_2 \parallel \frac{1}{Cs}$$

$$= \frac{R_2 / Cs}{R_2 + 1/Cs}$$

$$= \frac{R_2}{R_2 Cs + 1}$$

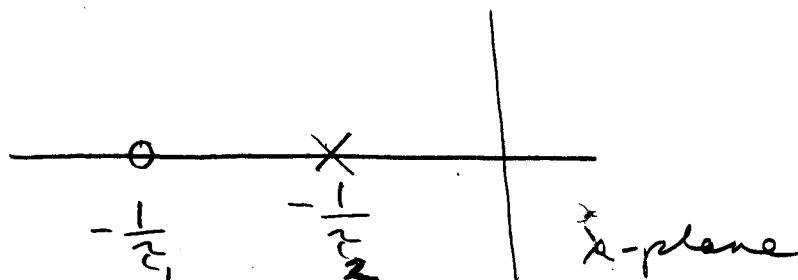
and thus

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_1 + Z_2}{Z_1} = \frac{R_1 + \frac{R_2}{R_2 Cs + 1}}{R_1}$$

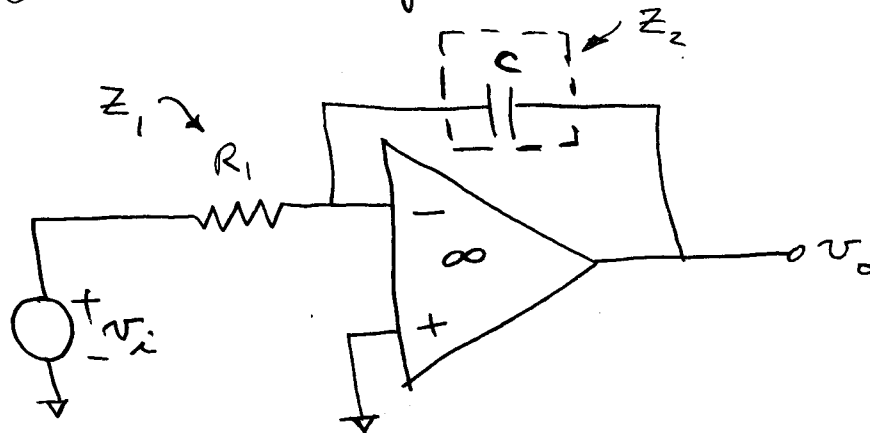
$$= \frac{R_1 R_2 Cs + R_1 + R_2}{R_1 (R_2 Cs + 1)} = \frac{R_1 + R_2}{R_1} \frac{\frac{R_1 R_2 Cs + 1}{R_1 + R_2}}{R_2 Cs + 1}$$

$$= K \frac{\tau_1 s + 1}{\tau_2 s + 1} \quad \text{where} \quad \begin{cases} K = (R_1 + R_2)/R_1 \\ \tau_1 = \frac{R_1 R_2}{R_1 + R_2} C \\ \tau_2 = R_2 C \end{cases}$$

Note that $\tau_1 < \tau_2$



Another example:



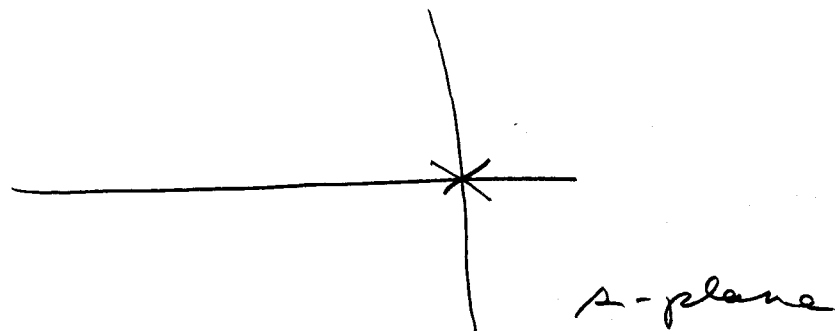
$$Z_1 = R_1$$

$$Z_2(s) = 1/sC$$

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{1/sC}{R_1} = -\frac{1}{RCs}$$

This is an inverting integrator

↑
pole at
 $s=0$



This is a useful building block for analog control circuitry.