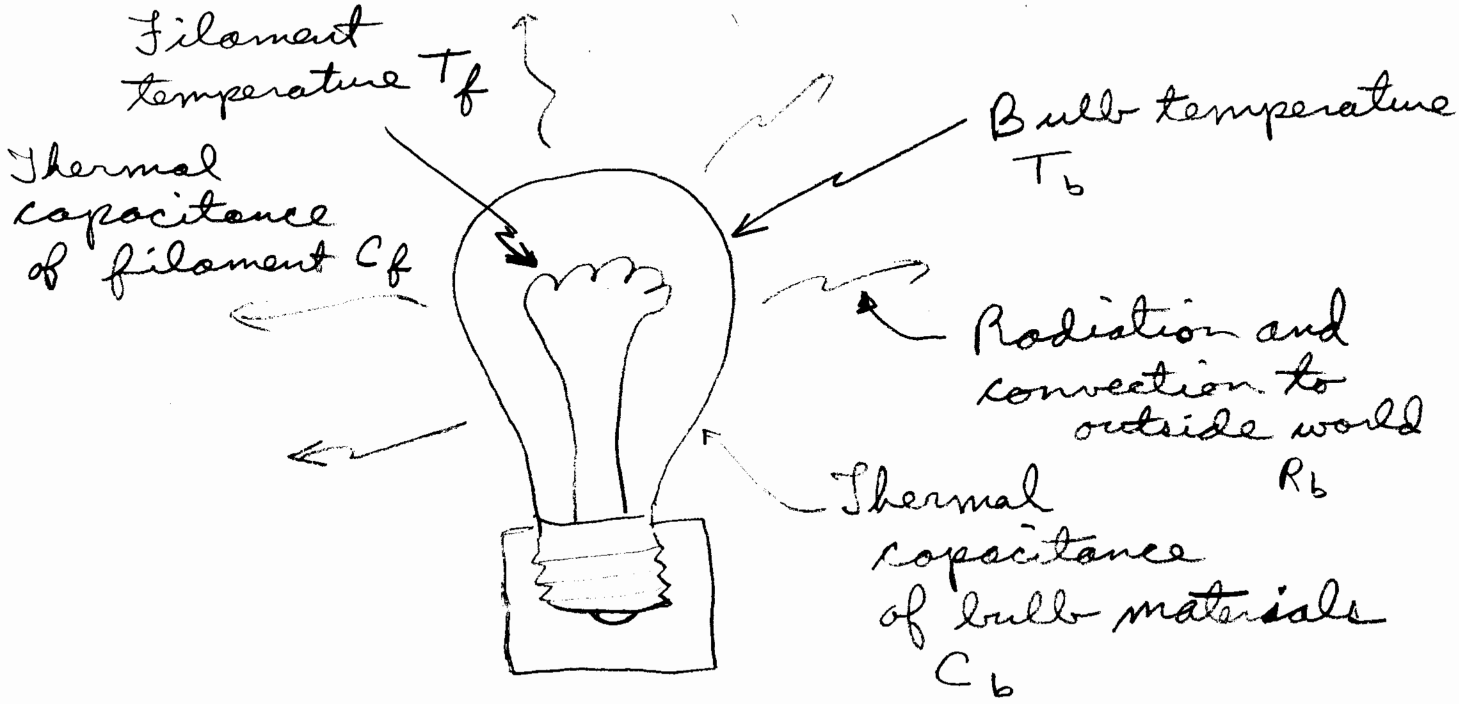
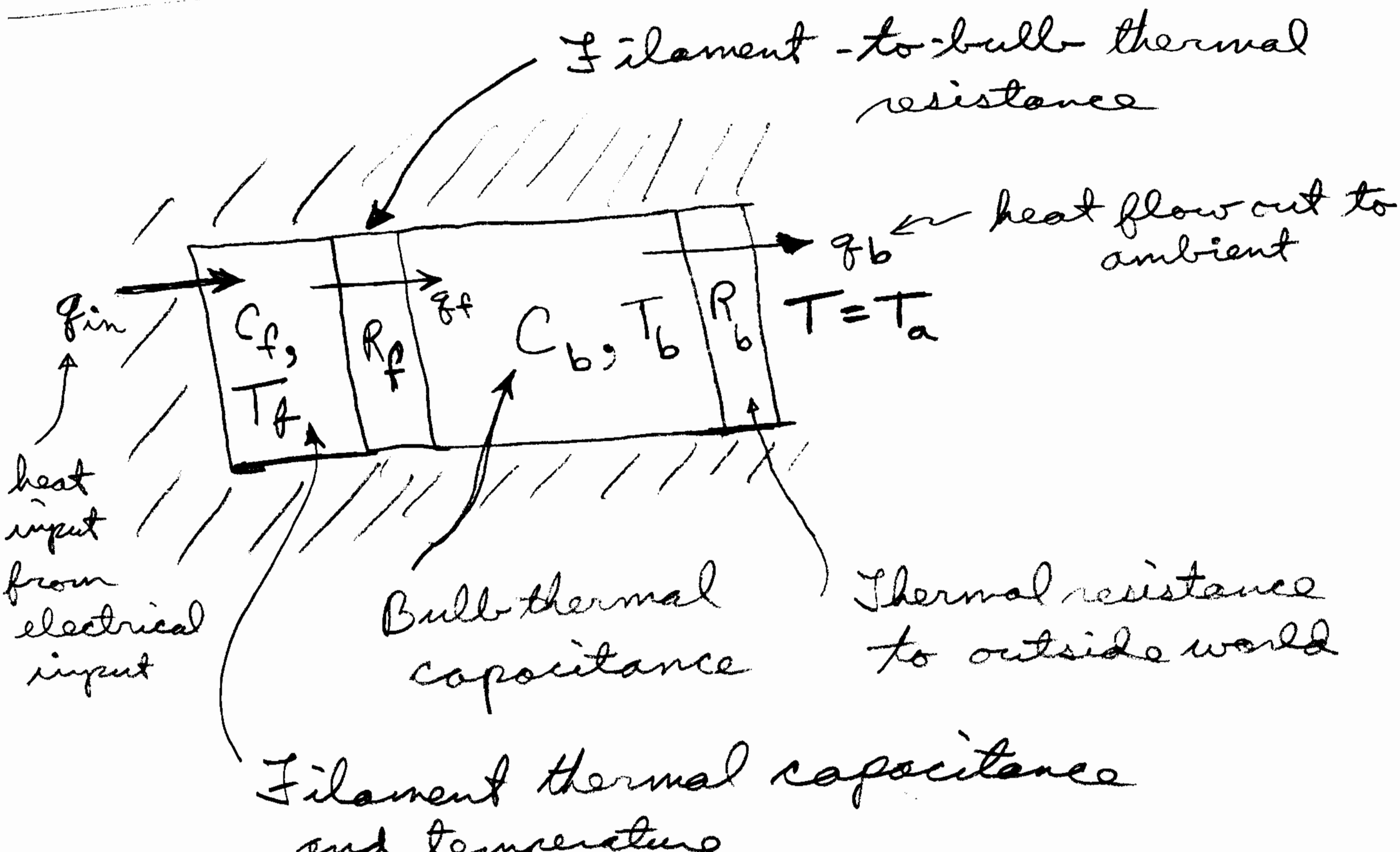
T1

We previously studied the temperature variation of a light bulb with a first-order model of bulb temperature. To gain additional insight, suppose we also include a model of the filament temperature inside the bulb. This requires adding an additional thermal mass C_f with its own temperature to represent the filament temperature and heat energy storage. These elements augment the model which previously only accounted for the temperature on the outside of the bulb T_b and thermal capacitance C_b.

We take some liberties here with the 4th-power radiative heat transfer relationship and assume that heat flow from the filament to the bulb can be represented by the linear thermal resistance R_f.

This augmented model is shown schematically at the top of the next page, and in lumped thermal element form at the bottom of the next page.


Filament temperature T_f
Thermal capacitance of filament C_f
Bulb temperature T_b
Radiation and convection to outside world R_b
Thermal capacitance of bulb materials C_b


Filament-to-bulb thermal resistance
heat flow out to ambient
q_b
q_{in}
C_f, T_f
R_f
q_f
C_b, T_b
R_b
$T = T_a$
heat input from electrical input
Bulb thermal capacitance
Thermal resistance to outside world
Filament thermal capacitance and temperature

Here the relevant temperature variables are

T_f = filament temperature [°K]

T_b = bulb-envelope temperature [°K]

T_a = ambient temperature [°K]

We also show the heat flows through the system:

q_{in} = heat input from filament [W]
(supplied from electrical source)

q_f = heat flow from filament to bulb [W]

q_b = heat flow from bulb to ambient [W]

Note that heat flows into the system from electrical heating of the filament, and out into ambient temperature.

The thermal capacitances C_f and C_b represent heat storage in the filament and bulb envelope, respectively. These have units of [J/°K].

The thermal resistances R_f and R_b represent the heat-flow connections between the filament/envelope and envelope/ambient, respectively. These have units of [°C/W].

The heat flows q_f and q_b represent the power flow in [W] through the thermal resistances.

The constitutive relations for the elements are

temperature of capacitance

Capacitance: $$C \frac{dT}{dt} = q$$

heat flow into capacitance

Resistance: $$T = qR$$

temperature drop across resistance

heat flow through resistance

To solve thermal problems, we will sum heat flows at the the thermal capacitances. This yields:

$$C_f \frac{dT_f}{dt} = q_{in} - q_f$$

$$C_b \frac{dT_b}{dt} = q_F - q_b$$

Now the constitutive relationships for the resistors give:

$$q_f = \frac{1}{R_f}(T_f - T_b)$$

$$q_b = \frac{1}{R_b}(T_b - T_a)$$

Combining these with the earlier pair of equations yields

$$C_f \frac{dT_f}{dt} = -\frac{1}{R_f}(T_f - T_b) + q_{in} \qquad (1)$$

$$C_b \frac{dT_b}{dt} = \frac{1}{R_f}(T_f - T_b) - \frac{1}{R_b}(T_b - T_a) \quad (2)$$

Equations (1) and (2) give a <u>state-space</u> model of the system. The state variables are the two temperatures T_f and T_b which represent storage of heat energy in the system. This form of model can be solved directly to find the initial condition response of the system, given the initial values $T_f(0)$ and $T_b(0)$. (We will discuss state-space later.)

An alternate representation is to transform (1) and (2) into a single 2nd-order differential equation. This representation is ommenable to solution via the techniques presented in this chapter.

In order to transform (1),(2) into a second-order ODE, we need to choose a variable to represent the system. Let's focus on the bulb envelope temperature T_b. Thus, we need to eliminate T_f from (1) and (2).

To facilitate this, it is helpful to introduce <u>operator</u> notation. That is the operator D represents differentiation

$$D f(t) \triangleq \frac{df(t)}{dt} = f'(t)$$

$$D^2 f(t) \triangleq \frac{d^2 f(t)}{dt} = f''(t)$$

and so forth.

With this notation, and some rearranging, equation (1) becomes

$$(R_f C_f D + 1) T_f = T_b + R_f q_{in}$$

and thus, taking the inverse operation on both sides

$$T_f = (R_f C_f D + 1)^{-1} (T_b + R_f q_{in})$$

Substitution of this result into (2) yields

$$\left[R_bC_bD + 1 + \frac{R_b}{R_f} - \frac{R_b}{R_f}(R_fC_fD+1)^{-1}\right]T_b$$

$$= R_b(R_fC_fD+1)^{-1}q_{in} + T_a$$

Now we multiply through by the operator (R_fC_fD+1) to give

$$\left[\left(R_bC_bD + 1 + \frac{R_b}{R_f}\right)(R_fC_fD+1) - \frac{R_b}{R_f}\right]T_b$$

$$= R_bq_{in} + (R_fC_fD+1)T_a$$

and thus

$$\left[R_bC_bR_fC_fD^2 + (R_f+R_b)C_fD + R_bC_bD + 1\right]T_b$$

$$= R_bq_{in} + (R_fC_fD+1)T_a$$

Convert back into explicit differential equation form to give

$$R_bC_bR_fC_f\frac{d^2T_b}{dt^2} + ((R_f+R_b)C_f + R_bC_b)\frac{dT_b}{dt} + T_b$$

$$= R_bq_{in} + R_fC_f\frac{dT_a}{dt} + T_a$$

a standard 2nd-order diff. eqn. inputs input

Now, we view the ambient temperature as a constant, and thus $\frac{dT_a}{dt} = 0$.

Further, it is useful to define the temperature <u>difference</u> $T \triangleq T_b - T_a$,

and thus $\frac{dT}{dt} = \frac{dT_b}{dt}$, $\frac{d^2T}{dt^2} = \frac{d^2T_b}{dt^2}$

With $dT_a/dt = 0$, and rewriting in terms of T, we have

$$R_bC_bR_fC_f \frac{d^2T}{dt^2} + ((R_f + R_b)C_f + R_bC_b)\frac{dT}{dt} + T = R_b q_{in}$$

Now suppose we set the system parameters as

$R_b = 2\ [°K/W]$

$R_f = 40\ [°K/W]$

$C_b = 65\ (J/°C]$

$C_f = 10^{-3} [J/°C]$

(these have been picked to reasonably fit the response of a real 60W bulb)

With these parameters, note that $R_f >> R_b$, and thus the term $(R_f + R_b) \simeq R_f$, and the differential equation can be written approximately as

$$R_bC_bR_fC_f \frac{d^2T}{dt^2} + (R_fC_f + R_bC_b)\frac{dT}{dt} + T = R_b q_{in}$$

This can then be directly factored as

$$\left(R_b C_b \frac{dT}{dt} + 1\right)\left(R_f C_f \frac{dT}{dt} + 1\right) = R_b q_{in}$$

The characteristic equation associated with this differential equation is

$$\left(R_b C_b s + 1\right)\left(R_f C_f s + 1\right) = 0$$

Plugging in numbers gives $R_b C_b = 130 \text{ sec} \triangleq \tau_b$

$R_f C_f = 0.04 \text{ sec} \triangleq \tau_f$

Notice the two widely different time constants.

The long-time scale $\tau_b = 130$ sec is associated with the thermal transient on the bulb envelope. It takes on the order of $5\tau_b \simeq 10$ minutes for the bulb temperature to equilibrate.

The short time scale $\tau_a = 0.04$ sec is associated with the thermal transient of the filament. This small piece of coiled wire heats up/cools down far more rapidly than the glass bulb envelope. It takes on the order of $5\tau_a = 0.2$ sec for the filament temperature to equilibrate.

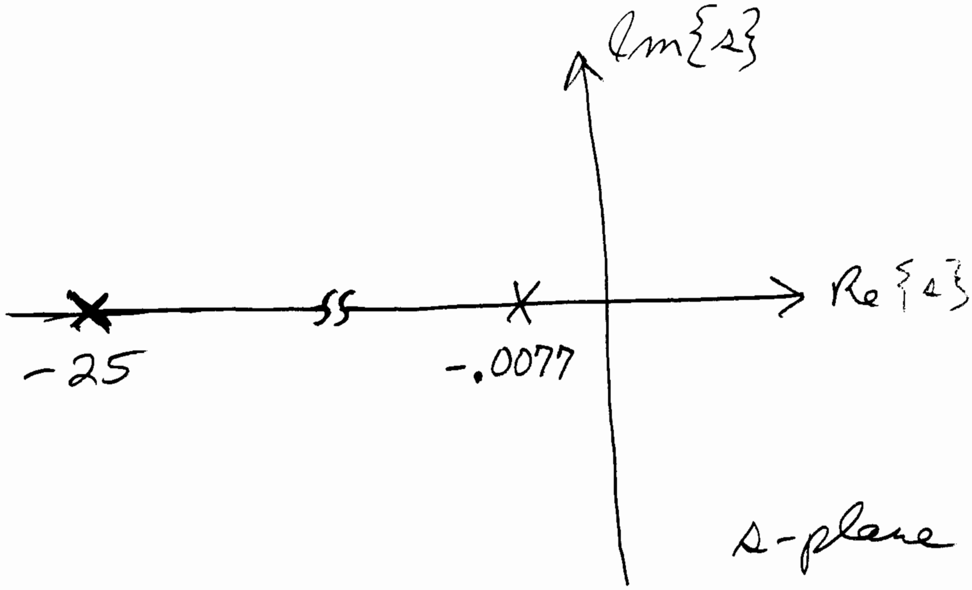
Note that this system is highly overdamped. If we set $\omega_n = \frac{1}{\sqrt{R_b C_b R_f C_f}}$ and $\frac{2\zeta}{\omega_n} = R_f C_f + R_b C_b$, we find that

$$\omega_n = 0.44 \text{ rad/sec} ; \quad \zeta = 57$$

The system poles are at

$$s_b = -\frac{1}{\tau_b} = -7.7 \times 10^{-3} \text{ sec}^{-1}$$

$$s_f = -\frac{1}{\tau_f} = -25 \text{ sec}^{-1}$$



And the natural response will be of the form

$$T(t) = c_1 e^{-t/\tau_b} + c_2 e^{-t/\tau_f}$$