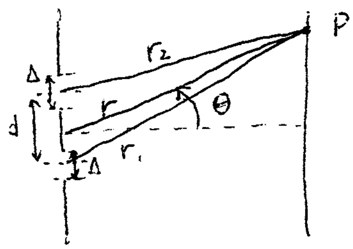
22.02 Fall 2006 Problem Set #1 Solutions

1. Assumptions: $r >> d$

Scalar wave

$\lambda >> \Delta$



Each slit is a source of a point radiation emitting a circular wave with a general form for amplitude: $\phi = \frac{\Delta}{r} A\cos(kr - \omega t)$ where ϕ = amplitude of circular wave after the slit, A = amplitude hitting the slit from the left, k = wave number $= \frac{2\pi}{\lambda}$, $\omega = 2\pi f = \frac{2\pi c}{\lambda}$

$r >> d \Rightarrow r_1 = r + \frac{d}{2}\sin\theta$, $r_2 = r - \frac{d}{2}\sin\theta$

The waves from the two slits have the amplitudes:

$$\Phi_1 = \frac{\Delta}{r_1} A\cos(kr_1 - \omega t) \approx \frac{\Delta}{r} A\cos(kr - \omega t + \tfrac{kd}{2}\sin\theta)$$

$$\Phi_2 = \frac{\Delta}{r_2} A\cos(kr_2 - \omega t) \approx \frac{\Delta}{r} A\cos(kr - \omega t - \tfrac{kd}{2}\sin\theta)$$

The total amplitude at P is the superposition of the two waves:

$$\Phi_p = \Phi_1 + \Phi_2 = \frac{\Delta}{r} A\left[\cos(kr - \omega t + \tfrac{kd}{2}\sin\theta) + \cos(kr - \omega t - \tfrac{kd}{2}\sin\theta)\right]$$

Trigonometric identity: $\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

$$\Phi_p = \frac{2\Delta A}{r}\cos(kr - \omega t)\cos(\tfrac{kd}{2}\sin\theta), \quad k = \frac{2\pi}{\lambda}, \quad \omega = \frac{2\pi c}{\lambda}$$

$$\langle \Phi_p^2 \rangle = \frac{1}{T}\int_0^T \Phi_p^2\, dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \Phi_p^2\, dt = \frac{1}{2}\left[\frac{2\Delta A}{r}\cos(\tfrac{kd}{2}\sin\theta)\right]^2$$

$$\langle \Phi_p^2 \rangle = \frac{2\Delta^2 A^2}{r^2}\cos^2(\tfrac{kd}{2}\sin\theta), \quad k = \frac{2\pi}{\lambda} \Rightarrow \boxed{\langle \Phi_p^2 \rangle = \frac{2\Delta^2 A^2}{r^2}\cos^2(\tfrac{\pi d}{\lambda}\sin\theta)}$$

2. $y = 0.3\sin[\pi(0.5x - 50t)]$

General form: $y(x,t) = A\sin[2\pi(\frac{x}{\lambda} + \frac{t}{T})]$, where A = amplitude, λ = wavelength, T = period

Rewrite in general form: $y = 0.3\sin[2\pi(\frac{x}{4} - \frac{t}{0.04})] \Rightarrow$

$\boxed{A = 0.3 \text{ cm},\ \lambda = 4 \text{ cm},\ T = 0.04 \text{s}}$

wave number $= k = \frac{2\pi}{\lambda} \Rightarrow \boxed{k = \frac{\pi}{2}\ \text{cm}^{-1}}$ frequency $= f = \frac{1}{T} \Rightarrow \boxed{f = 25 \text{ Hz}}$

velocity $= v = \frac{\lambda}{T} \Rightarrow \boxed{v = 100 \text{ cm/s}}$

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Maximize transverse velocity, $v(x,t)$:

$$v(x,t) = \frac{\partial}{\partial t}\left[y(x,t)\right] = -\pi(50)(0.3)\cos[\pi(0.5x-50t)] = -15\pi\cos[\pi(0.5x-50t)]$$

Maximum of $v(x,t)$ occurs when the cosine term equals -1 $\Rightarrow$

$\boxed{\text{Maximum transverse speed} = 15\pi\ \text{cm/s}}$

3. Coulombic potential energy: $V_{coul} = \frac{Z_1 Z_2 e^2}{r}$

$Z_1 = Z_{alpha} = 2$, $Z_2 = Z_{gold} = 79$

$r = 1.25 \times 200^{1/3}\ F = 7.31\ F$

$$e^2 = \frac{\hbar c}{137} = \frac{197}{137}\ MeV \cdot F = 1.43796\ MeV \cdot F$$

$$V_{coul} = \frac{2 \cdot 79 \cdot 1.43796\ MeV \cdot F}{7.31\ F} = 31.08\ MeV \Rightarrow \boxed{V_{coul} = 31.1\ MeV}$$

4. $v = \frac{c}{10} = 0.1c$

Non-relativistic: Kinetic Energy $= E_{kin} = \frac{1}{2}mv^2$

Neutron Mass $= m = 939.6\ MeV/c^2$

$$E_{kin} = \frac{1}{2}(939.6\ MeV/c^2)(0.1c^2) = 4.698\ MeV \Rightarrow \boxed{E_{kin} = 4.70\ MeV}$$

Relativistic: $E_{kin,rel} = m_0 c^2\left[\frac{1}{\sqrt{1-v^2/c^2}} - 1\right]$

$$E_{kin,rel} = (939.6\ MeV/c^2)c^2(0.005038) = 4.7335\ MeV$$

$\boxed{E_{kin,rel} = 4.73\ MeV}$

$E_{kin} \approx E_{kin,rel}$ (within 1%)

$\boxed{\text{Liboff : 3.1}}$

In general, for arbitrary operator $\hat{A}$, $\hat{A}^2\psi(x)$ implies: $\hat{A}[\hat{A}\psi(x)]$.

Using Table 3.1, we obtain the following:

- $\hat{D} = \frac{\partial}{\partial x}$; $\hat{D}\psi(x) = \frac{\partial \psi(x)}{\partial x}$ $\longrightarrow$ $\hat{D}^2 = \frac{\partial}{\partial x}\frac{\partial}{\partial x} = \frac{\partial^2}{\partial x^2}$; $\hat{D}^2\psi(x) = \frac{\partial}{\partial x}\left[\frac{\partial \psi(x)}{\partial x}\right] = \frac{\partial^2 \psi(x)}{\partial x^2}$

- $\hat{\Delta} = -\frac{\partial^2}{\partial x^2}$; $\hat{\Delta}\psi(x) = -\frac{\partial^2 \psi(x)}{\partial x^2}$; $\hat{\Delta}^2 = \left(-\frac{\partial^2}{\partial x^2}\right)\left(-\frac{\partial^2}{\partial x^2}\right) = \frac{\partial^4}{\partial x^4}$; $\hat{\Delta}^2\psi(x) = \frac{\partial^4 \psi(x)}{\partial x^4}$

- $\hat{M} = \frac{\partial^2}{\partial x \partial y}$; $\hat{M}\psi(x,y) = \frac{\partial^2 \psi(x,y)}{\partial x \partial y}$; $\hat{M}^2 = \frac{\partial^4}{\partial x^2 \partial y^2}$; $\hat{M}^2\psi(x,y) = \frac{\partial^4 \psi(x,y)}{\partial x^2 \partial y^2}$

- $\hat{I} =$ no change ; $\hat{I}\psi(x) = \psi(x)$; $\hat{I}^2 =$ operation that leaves ψ unchanged ; $\hat{I}^2\psi = \psi$

- $\hat{Q} = \int_0^1 dx'$; $\hat{Q}\psi(x) = \int_0^1 dx'\,\psi(x')$; $\hat{Q}^2 = \int_0^1 dx'' \int_0^1 dx'\,\psi(x') = \int_0^1 dx''\,\psi(x'') = \hat{Q}\psi(x) \longrightarrow \hat{Q}^2 = \hat{Q}$

- $\hat{F} = F(x)$; $\hat{F}\psi(x) = F(x)\psi(x)$; $\hat{F}^2 = F(x)\cdot F(x) = F^2(x)$; $\hat{F}^2\psi(x) = F^2(x)\,\psi(x)$

- $\hat{B} = \frac{1}{3}$; $\hat{B}\psi = \frac{1}{3}\psi$; $\hat{B}^2 = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}$; $\hat{B}^2\psi = \frac{1}{9}\cdot\psi$

- $\hat{\Theta} =$ operator that annihilates ψ ; $\hat{\Theta}\psi = 0$; $\hat{\Theta}^2 =$ operator that annihilates ψ ; $\hat{\Theta}^2\psi = \hat{\Theta}[\hat{\Theta}\psi] = \hat{\Theta}(0) = 0$

- $\hat{P} =$ operator that changes ψ to a specific polynomial ; $\hat{P}\psi = \psi^3 - 3\psi^2 - 4$; $\hat{P}^2\psi = \hat{P}(\hat{P}\psi) = \hat{P}(\psi^3 - 3\psi^2 - 4)$
  $= (\psi^3 - 3\psi^2 - 4)^3 - 3(\psi^3 - 3\psi^2 - 4)^2 - 4$

  $\therefore$ $\hat{P}^2 =$ operator that changes ψ to a specific polynomial of ψ

- $\hat{G} =$ operator that changes ψ to the number 8 ; $\hat{G}\psi = 8$; $\hat{G}^2\psi = \hat{G}(\hat{G}\psi) = \hat{G}(8) = 8$; $\hat{G}^2 =$ operator that changes ψ to number 8

1.5

$\boxed{\text{Liboff } 3.2}$

- $\hat{D} = \frac{\partial}{\partial x}$; $\hat{D}^{-1}\hat{D}\psi = \hat{D}^{-1}\left(\frac{\partial \psi}{\partial x}\right) = \psi$ Integration would yield constant of integration, ∴ $\hat{D}^{-1}$ does not exist, since we cannot invert constant.

- $\hat{I}^{-1}\hat{I}\psi = \hat{I}^{-1}\psi = \psi \longrightarrow \hat{I}^{-1} = \hat{I}$

- $\hat{F}^{-1}\hat{F}\psi = \hat{F}^{-1}[F(x)\psi] = \psi \longrightarrow \hat{F}^{-1} = \frac{1}{F(x)}$, if $F(x) \neq 0$
  For $F(x) = 0$, $\hat{F}^{-1}$ does not exist.

- $\hat{B}^{-1}\hat{B}\psi = \hat{B}^{-1}\left(\frac{1}{3}\psi\right) = \psi \longrightarrow \hat{B}^{-1} = 3$

- $\hat{\Theta}^{-1}\hat{\Theta}\psi = \hat{\Theta}^{-1}(0) = \psi \longrightarrow \hat{\Theta}^{-1}$ does not exist.

- $\hat{G}^{-1}\hat{G}\psi = \hat{G}^{-1}(8) = \psi \longrightarrow \hat{G}^{-1}$ does not exist.

Although this problem does not ask about $\hat{P}^{-1}$, note that it is a cubic root, but non-unique and very hard to define.

$\boxed{\text{Liboff 3.3}}$

- $\hat{D}(a\psi_1 + b\psi_2) = \frac{\partial}{\partial x}(a\psi_1 + b\psi_2) = a\frac{\partial \psi_1}{\partial x} + b\frac{\partial \psi_2}{\partial x} = a\hat{D}\psi_1 + b\hat{D}\psi_2$ ✓ $\rightarrow \hat{D}$ IS linear!

- $\hat{\Delta}(a\psi_1 + b\psi_2) = -\frac{\partial^2}{\partial x^2}(a\psi_1 + b\psi_2) = -a\frac{\partial^2 \psi_1}{\partial x^2} - b\frac{\partial^2 \psi_2}{\partial x^2} = a\hat{\Delta}\psi_1 + b\hat{\Delta}\psi_2$ ✓ $\rightarrow \hat{\Delta}$ IS linear!

- $\hat{M}(a\psi_1 + b\psi_2) = \frac{\partial^2}{\partial x \partial y}(a\psi_1 + b\psi_2) = a\frac{\partial^2 \psi_1}{\partial x \partial y} + b\frac{\partial^2 \psi_2}{\partial x \partial y} = a\hat{M}\psi_1 + b\hat{M}\psi_2$ ✓ $\rightarrow \hat{M}$ IS linear!

- $\hat{I}(a\psi_1 + b\psi_2) = a\psi_1 + b\psi_2 = a\hat{I}\psi_1 + b\hat{I}\psi_2$ ✓ $\rightarrow \hat{I}$ IS linear!

- $\hat{Q}(a\psi_1 + b\psi_2) = \int_0^1 (a\psi_1 + b\psi_2)\,dx' = a\int_0^1 \psi_1\,dx' + b\int_0^1 \psi_2\,dx' = a\hat{Q}\psi_1 + b\hat{Q}\psi_2$ ✓ $\rightarrow \hat{Q}$ IS linear!

- $\hat{F}(a\psi_1 + b\psi_2) = F(x)[a\psi_1 + b\psi_2] = aF(x)\psi_1 + bF(x)\psi_2 = a\hat{F}\psi_1 + b\hat{F}\psi_2$ ✓ $\rightarrow \hat{F}$ IS linear!

- $\hat{B}(a\psi_1 + b\psi_2) = \frac{1}{3}[a\psi_1 + b\psi_2] = \frac{1}{3}a\psi_1 + \frac{1}{3}b\psi_2 = a\hat{B}\psi_1 + b\hat{B}\psi_2$ ✓ $\rightarrow \hat{B}$ IS linear!

- $\hat{\Theta}(a\psi_1 + b\psi_2) = 0 = a\cdot 0 + b\cdot 0 = a\hat{\Theta}\psi_1 + b\hat{\Theta}\psi_2$ ✓ $\rightarrow \hat{\Theta}$ IS linear!

- $\hat{P}(a\psi_1 + b\psi_2) = [a\psi_1 + b\psi_2]^3 - 3[a\psi_1 + b\psi_2]^2 - 4 \neq a\hat{P}\psi_1 + b\hat{P}\psi_2 = a\psi_1^3 - 3a\psi_1^2 - 4a + b\psi_2^3 - 3b\psi_2^2 - 4b$

  $\rightarrow \hat{P}$ IS NOT linear!

- $\hat{G}(a\psi_1 + b\psi_2) = 8 \neq a\hat{G}\psi_1 + b\hat{G}\psi_2 = 8a + 8b$

  $\hat{G}$ IS NOT linear!

$\boxed{\text{Liboff } 3.6}$

The goal of this problem is to verify the identities of delta function by employing a test function. In general, we have LHS = RHS or $D_1(y) = D_2(y)$.

For equality to hold, for all "ordinary functions" $F(y)$, the following must be true:

$$\int_{-\infty}^{\infty} F(y)\, D_1(y)\, dy = \int_{-\infty}^{\infty} F(y)\, D_2(y)\, dy$$

a) $\delta(y) = \delta(-y)$: Make a change of variables → Let $t = -y$, then $dt = -dy$

$$\rightarrow \int_{-\infty}^{+\infty} F(y)\,\delta(y)\,dy = \int_{+\infty}^{-\infty} \delta(t) F(-t)(-1)dt = -\int_{+\infty}^{-\infty} F(-t)\,\delta(t)\,dt = \int_{-\infty}^{+\infty} F(-t)\,\delta(t)\,dt = F(0)$$

$$LHS = \int_{-\infty}^{\infty} F(y)\delta(y)dy = F(0), \quad \therefore\ LHS = RHS \ \checkmark \quad \boxed{\delta(y) = \delta(-y)}$$

b) $\delta'(y) = -\delta'(-y)$ Integrate by parts

$$LHS = \int_{-\infty}^{+\infty} F(y)\delta'(y)dy = \int_{-\infty}^{+\infty} \frac{d}{dy}[\delta(y)F(y)]dy - \int_{-\infty}^{+\infty} F'(y)\,\delta(y)\,dy$$

$$= 0 - F'(0) = -F'(0)$$

RHS : Make same change of variables as (a): $t = -y$
→ Integrate by parts.

$$RHS = -\int_{-\infty}^{+\infty} F(y)\,\delta'(-y)\,dy = -\int_{+\infty}^{-\infty} F(-t)\,\delta'(t)\cdot(-1)dt = -\int_{-\infty}^{+\infty} F(-t)\,\delta'(t)\,dt$$

$$= -\left[\int_{-\infty}^{+\infty} \frac{d}{dt}[F(-t)\,\delta(t)]dt - \int_{-\infty}^{+\infty} \frac{d}{dt}[F(-t)]\,\delta(t)\,dt\right]$$

$$= 0 - F'(0) = -F'(0) \quad \rightarrow LHS = RHS \ \therefore \boxed{\delta'(y) = -\delta'(-y)}$$

c) $y\,\delta(y) = 0$ → Just integrate

$$LHS = \int_{-\infty}^{+\infty} y F(y)\,\delta(y)\,dy = [yF(y)]\Big|_{y=0} = 0 \qquad LHS = RHS \ \therefore \boxed{y\delta(y) = 0}$$

d) $\delta(ay) = |a^{-1}|\,\delta(y)$ This is general case of part (a). This time, let $t = ay$

$$\text{For } a>0: \ LHS = \int_{-\infty}^{+\infty} F(y)\,\delta(ay)\,dy = \int_{-\infty}^{\infty} F(t/a)\,\delta(t)\,\tfrac{1}{a}\,dt = a^{-1}F(0) = a^{-1}\int_{-\infty}^{\infty} F(y)\delta(y)\,dy$$

$$\text{For } a<0, \ LHS = \int_{-\infty}^{+\infty} F(y)\,\delta(ay)dy = \int_{+\infty}^{-\infty} F(t/a)\,\delta(t)\,\tfrac{1}{a}\,dt = -a^{-1}\int_{-\infty}^{+\infty} F(t/a)\,\delta(t)\,dt$$

$$= -a^{-1}F(0) = |a|^{-1}F(0) = |a|^{-1}\int_{-\infty}^{\infty} F(y)\delta(y)\,dy \ \checkmark \quad \boxed{\delta(ay) = |a^{-1}|\,\delta(y)}$$

e) $\delta(y^2-a^2) = |2a|^{-1}[\delta(y-a) + \delta(y+a)]$ Change variables: Let $t = y^2 - a^2$, then $dt = 2y\,dy = 2(\pm\sqrt{t+a^2})dy$
where + sign is for $y>0$ and − sign is for $y<0$

$$\text{Now, } LHS = \int_{-\infty}^{0} F(y)\,\delta(y^2-a^2)\,dy + \int_{0}^{+\infty} F(y)\,\delta(y^2-a^2)dy = -\int_{+\infty}^{-a^2} \frac{F(-\sqrt{t+a^2})}{2\sqrt{t+a^2}}\,\delta(t)dt + \int_{-a^2}^{+\infty} \frac{F(+\sqrt{t+a^2})}{2\sqrt{t+a^2}}\,\delta(t)dt$$

$$= \frac{1}{|2a|}F(-a) + \frac{1}{|2a|}F(+a) \ \checkmark \quad \therefore \boxed{\delta(y^2-a^2) = \frac{1}{|2a|}[\delta(y-a) + \delta(y+a)]}$$

continued →

(Liboff 3.6 continued)

f) $\int_{-\infty}^{+\infty} \delta(a-y)\,\delta(y-b)\,dy = \delta(a-b)$

Here, let $\delta(a-y) = f(y)$. Then $\int_{-\infty}^{+\infty} f(y)\,\delta(y-b)\,dy = f(b)$

Now, substituting for $f(y=b)$, we have $f(b) = \delta(a-b) = RHS$ ✓

$$\boxed{\int_{-\infty}^{+\infty} \delta(a-y)\,\delta(y-b)\,dy = \delta(a-b)}$$

g) $f(y)\,\delta(y-a) = f(a)\,\delta(y-a)$ → Integrate LHS, then RHS

$LHS = \int_{-\infty}^{+\infty} f(y)\,\delta(y-a)\,dy = f(a)$

$RHS = \int_{-\infty}^{+\infty} f(a)\,\delta(y-a) = f(a)$

$\Big\}$ LHS = RHS ✓ $\boxed{f(y)\,\delta(y-a) = f(a)\,\delta(y-a)}$

h) $y\,\delta'(y) = -\delta(y)$ This was done as an example in Liboff.

See p.75, equations (3.30) and (3.31)

i) $\delta[g(y)] = \sum_{y_i} \frac{\delta(y-y_i)}{|g'(y_i)|}$, $g(y_i)=0$

This requires a Taylor expansion. First, consider $LHS = \int_{-\infty}^{+\infty} f(y)\,\delta[g(y)]\,dy$

Assume y_i are zeros of $g(y)$ (since we are told $g(y_i)=0$), and suppose $g'(y_i) \neq 0$. Near each y_i, expand $g(y) \cong g'(y_i)\cdot(y-y_i)$.

Now, the integral can be broken into disjoint pieces that bracket each y_i.

Use series expansion and results of part (d) to get: $\int_{-\infty}^{+\infty} f(y)\,\delta[g(y)]\,dy = \sum_{y_i} \frac{f(y_i)}{|g'(y_i)|}$,

if $g'(y_i) \neq 0$. Now, we conclude $\delta[g(y_i)] = \sum_{y_i} \frac{\delta(y-y_i)}{|g'(y_i)|}$, since

$\sum_{y_i} \frac{\delta(y-y_i)}{|g'(y_i)|}$ behaves the same way under the integral [that is, $f(y_i) = \delta(y-y_i)$]

$$\boxed{\delta[g(y)] = \sum_{y_i} \frac{\delta(y-y_i)}{|g'(y_i)|}}$$

Note that a change of variables would work too.

Let $t = g(y)$.