22.02 Fall 2006 Problem Set #2 Solutions

1. (Liboff 3.11) $\Delta P = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$, $\langle P \rangle = \int \psi^* \hat{p} \psi \, dx$, $\langle P^2 \rangle = \int \psi^* \hat{p^2} \psi \, dx$

$$\psi(x,t) = A\exp\left[\frac{-(x-x_0)^2}{4a^2}\right]\exp\left(\frac{iP_0 x}{\hbar}\right)\exp(-i\omega_0 t) = A\exp\left[\frac{-(x-x_0)^2}{4a^2} + \frac{iP_0 x}{\hbar} - i\omega_0 t\right]$$

$\hat{P} = -i\hbar \frac{\partial}{\partial x}$ (momentum operator), $\hat{p^2} = -\hbar^2 \frac{\partial^2}{\partial x^2}$

$$\langle P \rangle = \int_{-\infty}^{\infty} \psi^*(-i\hbar)\left[\frac{-(x-x_0)}{2a^2} + P_0\left(\frac{i}{\hbar}\right)\right]\psi \, dx = \int_{-\infty}^{\infty} A^2\left[\frac{i\hbar(x-x_0)}{2a^2} + P_0\right]\exp\left[\frac{-(x-x_0)^2}{2a}\right]$$

let $\eta = \frac{x-x_0}{a} \Rightarrow dx = a\,d\eta \Rightarrow \langle P \rangle = A^2 a \int_{-\infty}^{\infty}\left[P_0 + \frac{i\hbar\eta}{2a}\right]\exp\left(\frac{-\eta^2}{2}\right)d\eta$

$$\langle P \rangle = A^2 a\left[\int_{-\infty}^{\infty} P_0 \exp\left(\frac{-\eta^2}{2}\right)d\eta + \int_{-\infty}^{\infty}\frac{i\hbar\eta}{2a}\exp\left(\frac{-\eta^2}{2}\right)d\eta\right]$$

$\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}} \Rightarrow \int_{-\infty}^{\infty} P_0 \exp\left(\frac{-\eta^2}{2}\right)d\eta = P_0\sqrt{2\pi}$; $\eta\exp\left(\frac{-\eta^2}{2}\right)$ is odd $\Rightarrow \int_{-\infty}^{\infty}\frac{i\hbar\eta}{2a}\exp\left(\frac{-\eta^2}{2}\right)d\eta = 0$

$\langle P \rangle = A^2 a P_0 \sqrt{2\pi}$, normalize to find A $\left(\int_{-\infty}^{\infty}\psi^*\psi\,dx = 1\right)$

$$1 = A^2\int_{-\infty}^{\infty}\exp\left[\frac{-(x-x_0)^2}{2a^2}\right]dx = A^2 a\int_{-\infty}^{\infty}\exp\left(\frac{-\eta^2}{2}\right)d\eta = A^2 a\sqrt{2\pi} \Rightarrow A^2 = \frac{1}{a\sqrt{2\pi}} \Rightarrow \boxed{\langle P \rangle = P_0}$$

$$\langle P^2 \rangle = \int_{-\infty}^{\infty}\psi^*(-\hbar^2)\frac{\partial^2\psi}{\partial x^2}dx = -\hbar^2\int_{-\infty}^{\infty}\psi^*\frac{\partial}{\partial x}\left\{\left[\frac{-(x-x_0)}{2a^2} + \frac{iP_0}{\hbar}\right]\psi\right\}dx$$

$$\langle P^2 \rangle = -\hbar^2\int_{-\infty}^{\infty}\psi^*\left\{\left[\frac{-(x-x_0)}{2a^2} + \frac{iP_0}{\hbar}\right]^2\psi + \psi\left(-\frac{1}{2a^2}\right)\right\}dx = -\hbar^2\int_{-\infty}^{\infty}\left\{\left[\frac{-(x-x_0)}{2a^2} + \frac{iP_0}{\hbar}\right]^2 - \frac{1}{2a^2}\right\}A^2\exp\left[\frac{-(x-x_0)^2}{2a^2}\right]dx$$

$$\langle P^2 \rangle = -\hbar^2 A^2 a\int_{-\infty}^{\infty}\left[\frac{\eta^2}{4a^2} - \frac{iP_0\eta}{2a\hbar} - \frac{P_0^2}{\hbar^2} - \frac{1}{2a^2}\right]\exp\left(\frac{-\eta^2}{2}\right)d\eta$$

$$\int_{-\infty}^{\infty}x^2 e^{-\alpha x^2}dx = \frac{\sqrt{\pi}}{2\alpha^{3/2}} \Rightarrow \langle P^2 \rangle = -\hbar^2 A^2 a\left[\frac{1}{4a^2} - 0 - \frac{P_0^2}{\hbar^2} - \frac{1}{2a^2}\right]\sqrt{2\pi}$$

$$A^2 = \frac{1}{a\sqrt{2\pi}} \Rightarrow \langle P^2 \rangle = P_0^2 + \frac{\hbar^2}{4a^2}, \quad \Delta P = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{\hbar^2}{4a^2}} = \pm\frac{\hbar}{2a} \Rightarrow \boxed{\Delta P = \pm\frac{\hbar}{2a}}$$

$\Delta X = a$ (from Liboff problem 3.10) $\Rightarrow \Delta X \Delta p = \frac{\hbar}{2}$ (consistent with Heisenberg Uncertainty Principle)

(Liboff 3.12) (a) Discrete results: $\langle C \rangle = \sum_{\text{all } C} c_i P(c_i)$

Randomly thrown die can have six outcomes: 1, 2, 3, 4, 5, 6 all with a $\frac{1}{6}$ probability

$$\langle s \rangle = \sum_{i=1}^{6} s_i P(s_i) = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = 3.5 \Rightarrow \boxed{\langle s \rangle = 3.5}$$

(b) $\langle s^2 \rangle = \sum_{i=1}^{6} s_i^2 P(s_i) = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{91}{6} = 15.167$

$$\Delta S = \sqrt{15.167 - 3.5^2} = 1.70783 \Rightarrow \boxed{\Delta S = 1.708}$$

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2. $\hat{H}\Psi_n = E_n\Psi_n$, $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ (Hamiltonian) $\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi_n = E_n\Psi_n$, $0<x<L$ (inside the box)

Boundary conditions: $\Psi_n(x=0)=0$, $\Psi_n(x=L)=0$

$$\frac{\partial^2\Psi_n}{\partial x^2} + \frac{2mE_n}{\hbar^2}\Psi_n = 0 \text{ , let } k_n^2 = \frac{2mE_n}{\hbar^2} \Rightarrow \frac{\partial^2\Psi_n}{\partial x^2} + k_n^2\Psi_n = 0$$

The general solution to this differential equation is: $\Psi_n(x) = A\sin(k_n x) + B\cos(k_n x)$

Apply the first boundary condition: $\Psi_n(x=0)=0 \Rightarrow A[\sin(0)] + B[\cos(0)] = B = 0$

$\Psi_n(x) = A\sin(k_n x)$

Apply the second boundary condition: $\Psi_n(x=L)=0 \Rightarrow A[\sin(k_n L)] = 0$

A cannot be zero (or $\Psi_n = 0$) $\Rightarrow \sin(k_n L) = 0 \Rightarrow k_n L = n\pi$, $n=1,2,3,\ldots \Rightarrow \Psi_n = A\sin\left(\frac{n\pi x}{L}\right)$

Normalize Ψ_n to find A: $\int_{-\infty}^{\infty}|\Psi_n|^2 dx = 1 \Rightarrow \int_0^L A^2\sin^2\left(\frac{n\pi x}{L}\right)dx = 1 \Rightarrow A^2\left(\frac{L}{2}\right) = 1 \Rightarrow A = \sqrt{\frac{2}{L}}$

$$\boxed{\begin{array}{ll} \Psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \text{ for } n=1,2,3,\ldots & 0 \le x \le L \\ \Psi_n(x) = 0 & x<0,\ x>L \end{array}}$$

$$k_n^2 = \frac{2mE_n}{\hbar^2} \Rightarrow E_n = \frac{\hbar^2 k_n^2}{2m} \text{ , } k_n = \frac{n\pi}{L} \Rightarrow \boxed{E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \text{ for } n=1,2,3,\ldots}$$

- $P(x) = |\Psi(x)|^2$, Probability that a position measurement finds the particle within Δx about x $= P(x)dx = |\Psi(x)|^2 dx$

$$\boxed{P(x)\,dx = \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right)\Delta x}$$

- $\Psi(x) = a_2\Psi_2(x) + a_5\Psi_5(x)$

a) The possible outcomes of an energy measurement are the energy eigenvalues. Because we have Ψ_2 & Ψ_5 the energy eigenvalues are E_2 & E_5 ($n=2$ & $n=5$).

$$\boxed{E_2 = \frac{2\pi^2\hbar^2}{mL^2} \text{ , } E_5 = \frac{25\pi^2\hbar^2}{2mL^2}}$$

b) $P(E_2) = P_2 = a_2^2$, $P(E_5) = P_5 = a_5^2 \Rightarrow a_2^2 = \frac{1}{4}$, $a_5^2 = \frac{3}{4} \Rightarrow \boxed{a_2 = \pm\frac{1}{2},\ a_5 = \pm\frac{\sqrt{3}}{2}}$

c) $|\Psi|^2 = |\Psi(x=\frac{L}{2})|^2 = |a_2\Psi_2(x=\frac{L}{2}) + a_5\Psi_5(x=\frac{L}{2})|^2 = \left|\pm\frac{1}{2}\sqrt{\frac{2}{L}}\sin(\pi) \pm \frac{\sqrt{3}}{2}\sqrt{\frac{2}{L}}\sin\left(\frac{5\pi}{2}\right)\right|^2$

$$\boxed{|\Psi(x=\tfrac{L}{2})|^2 = \frac{3}{2L}}$$

More specifically, the probability of finding the particle in Δx about $x = \frac{L}{2}$ is $P(x)\Delta x = \frac{3\Delta x}{2L}$.

3. <u>de Broglie wavelengths</u>

1 keV electron: $p = \hbar k$, $k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi\hbar}{p}$, $E = \frac{p^2}{2m} \Rightarrow \lambda = \frac{\sqrt{2}\,\pi\hbar}{\sqrt{mE}} = \frac{\sqrt{2}\,\pi\hbar c}{\sqrt{mc^2 E}}$

$m = m_{e^-} = 0.511\ MeV/c^2$, $\hbar c = 197\ MeV \cdot F$, $E = 1\,keV = 0.001\ MeV$

$\boxed{\lambda_{e^-} = 3.87 \times 10^4\ F = 3.87 \times 10^{-9}\ cm}$ λ_{e^-} is on the order of the size of an atom

4 MeV α-particle: $m = m_\alpha = 3728\ MeV/c^2$

$\boxed{\lambda_\alpha = 7.17\ F = 7.17 \times 10^{-13}\ cm}$ λ_α is on the order of the size of a nucleus

<u>Derive $\lambda = 2\pi \frac{\hbar c}{mc^2} \frac{c}{v}$</u>

$p = \hbar k$, $k = \frac{2\pi}{\lambda} \Rightarrow p = \frac{2\pi\hbar}{\lambda} \Rightarrow \lambda = \frac{2\pi\hbar}{p}$, $p = mv \Rightarrow \lambda = \frac{2\pi\hbar}{mv}$

$$\boxed{\lambda = 2\pi \frac{\hbar c}{mc^2} \frac{c}{v}}$$

<u>Chalk velocity</u>

$$\lambda = 2\pi\left(\frac{\hbar c}{mc^2}\right)\left(\frac{c}{v}\right) \Rightarrow v = 2\pi\left(\frac{\hbar c}{mc^2}\right)\left(\frac{c}{\lambda}\right)$$

$$m = 10^{24}\text{ carbon atoms}\left(\frac{12\text{ amu}}{\text{carbon atom}}\right)\left(\frac{931.5\ MeV/c^2}{\text{amu}}\right) = 1.12 \times 10^{28}\ MeV/c^2$$

$\lambda = 1\,cm = 0.01\,m$, $c = 2.9979 \times 10^8\ m/s$

$v = 3.31 \times 10^{-15}\ F/s \Rightarrow \boxed{v = 3.31 \times 10^{-30}\ m/s}$