22.02 Fall 2006 Problem Set #4 Solutions

1. Liboff, problem 5.12

(a) $[\hat{A}+\hat{B},\hat{C}] = (\hat{A}+\hat{B})\hat{C} - \hat{C}(\hat{A}+\hat{B}) = \hat{A}\hat{C}+\hat{B}\hat{C}-\hat{C}\hat{A}-\hat{C}\hat{B}$

$[\hat{A},\hat{C}]+[\hat{B},\hat{C}] = \hat{A}\hat{C}-\hat{C}\hat{A}+\hat{B}\hat{C}-\hat{C}\hat{B} = [\hat{A}+\hat{B},\hat{C}]$

(b) $[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$

$[\hat{A}\hat{B},\hat{C}] = (\hat{A}\hat{B})\hat{C} - \hat{C}(\hat{A}\hat{B})$

$\hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B} = \hat{A}(\hat{B}\hat{C}-\hat{C}\hat{B}) + (\hat{A}\hat{C}-\hat{C}\hat{A})\hat{B} = \hat{A}\hat{B}\hat{C}-\hat{A}\hat{C}\hat{B}+\hat{A}\hat{C}\hat{B}-\hat{C}\hat{A}\hat{B} =$

$(\hat{A}\hat{B})\hat{C} - \hat{C}(\hat{A}\hat{B}) = [\hat{A}\hat{B},\hat{C}]$

2. $\hat{P}[f(x)] = f(-x)\ ,\ \hat{H} = \hat{p}^2/2m = \left(\frac{-\hbar^2}{2m}\right)\frac{\partial^2}{\partial x^2}$

$[\hat{H},\hat{P}]f(x) = \hat{H}\{\hat{P}[f(x)]\} - \hat{P}\{\hat{H}[f(x)]\} = \hat{H}[f(-x)] - \hat{P}\left\{\frac{-\hbar^2}{2m}\frac{\partial^2[f(x)]}{\partial x^2}\right\}$

$= \left(\frac{-\hbar^2}{2m}\right)\left\{\frac{\partial^2[f(-x)]}{\partial x^2}\right\} - \left(\frac{-\hbar^2}{2m}\right)\left\{\frac{\partial^2[f(-x)]}{\partial x^2}\right\} = 0$

$\phi_1(x) = e^{ikx}\ ,\ \phi_2(x) = e^{-ikx}$

Parity eigenfunctions include even and odd functions such as sine & cosine

$\psi_1 = \frac{1}{2}e^{ikx} + \frac{1}{2}e^{-ikx} = \cos kx \Rightarrow T_{11} = \frac{1}{2}\ ,\ T_{12} = \frac{1}{2}$

$\psi_1 = \frac{1}{2i}e^{ikx} + \left(-\frac{1}{2i}\right)e^{-ikx} = \sin kx \Rightarrow T_{21} = \frac{1}{2i}\ ,\ T_{22} = -\frac{1}{2i}$

Eigenvalues:

$\hat{H}\psi_1 = a_{11}\psi_1 = -\frac{\hbar^2}{2m}\left(\frac{\partial^2(\cos kx)}{\partial x^2}\right) = \frac{+\hbar^2k^2\cos(kx)}{2m} \Rightarrow a_{11} = +\hbar^2k^2/2m$

$\hat{P}\psi_1 = a_{12}\psi_1 = \cos(-kx) = \cos(kx) \Rightarrow a_{12} = 1$

$\hat{H}\psi_2 = a_{21}\psi_2 = -\frac{\hbar^2}{2m}\left(\frac{\partial^2(\sin kx)}{\partial x^2}\right) = \frac{+\hbar^2k^2\sin(kx)}{2m} \Rightarrow a_{21} = +\hbar^2k^2/2m$

$\hat{P}\psi_2 = a_{22}\psi_2 = \sin(-kx) = -\sin(kx) \Rightarrow a_{22} = -1$

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3. $\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(x,y,z) = \hat{T} + \hat{V}$, $\hat{T} = \frac{\hat{p}^2}{2m} \Rightarrow [\hat{T},\hat{p}] = 0$

$[\hat{H},\hat{p}_x] = [(\hat{T}+\hat{V}),p_x] = [\hat{T},\hat{p}_x] + [\hat{V},\hat{p}_x] = [\hat{V},\hat{p}_x]$

$[\hat{H},\hat{p}_x]\psi = -i\hbar V \frac{\partial\psi}{\partial x} + i\hbar\frac{\partial}{\partial x}(V\psi) = i\hbar\left(-V\frac{\partial\psi}{\partial x} + V\frac{\partial\psi}{\partial x} + \psi\frac{\partial V}{\partial x}\right) \Rightarrow$

$[\hat{H},\hat{p}_x] = i\hbar\frac{\partial V}{\partial x}$; Similarly $[\hat{H},\hat{p}_y] = i\hbar\frac{\partial V}{\partial y}$ & $[\hat{H},\hat{p}_z] = i\hbar\frac{\partial V}{\partial z}$

(a) $V = ax + by^2 + cz^3 \Rightarrow \frac{\partial V}{\partial x} \neq 0$, $\frac{\partial V}{\partial y} \neq 0$, $\frac{\partial V}{\partial z} \neq 0 \Rightarrow$

No directions of momentum commute with Hamiltonian

(b) $V = a\arctan(x/L) \Rightarrow \frac{\partial V}{\partial x} \neq 0$, $\frac{\partial V}{\partial y} = 0$, $\frac{\partial V}{\partial z} = 0 \Rightarrow$

y & z directions of momentum commute with Hamiltonian

(c) $V = b\exp(-y^2/d^2) \Rightarrow \frac{\partial V}{\partial x} = 0$, $\frac{\partial V}{\partial y} \neq 0$, $\frac{\partial V}{\partial z} = 0 \Rightarrow$

x & z directions of momentum commute with Hamiltonian

4. $\hat{P}\psi(x,y,z) = \psi(-x,-y,-z)$

$[\hat{H},\hat{P}] = 0 \Rightarrow [(\hat{T}+\hat{V}),\hat{P}] = 0 \Rightarrow [\hat{T},\hat{P}] + [\hat{V},\hat{P}] = 0$

$[\hat{T}_x,\hat{P}_x] = 0$, $[\hat{T}_y,\hat{P}_y] = 0$, $[\hat{T}_z,\hat{P}_z] = 0$

$[\hat{V},\hat{P}]\psi(x,y,z) = \hat{V}[\hat{P}\psi(x,y,z)] - \hat{P}[\hat{V}\psi(x,y,z)] = V(x,y,z)[\psi(-x,-y,-z)] +$
$-\hat{P}[V(x,y,z)\psi(x,y,z)] = V(x,y,z)\psi(-x,-y,-z) - V(-x,-y,-z)\psi(-x,-y,-z)$

$[\hat{V},\hat{P}] = 0$ iff $V(x,y,z) = V(-x,-y,-z) \Rightarrow V$ must be an even function for $\hat{H}$ & $\hat{P}$ to commute and to share common eigenfunctions

(a) $V = e^2/r = e^2/\sqrt{x^2+y^2+z^2}$, V is even $\Rightarrow$ common eigenfunctions

(b) $V = ax + by^2 + cz^3$, V is not even $\Rightarrow$ no common eigenfunctions

(c) $V = axy$, V is even $\Rightarrow$ common eigenfunctions