22.02 Fall 2006 Problem Set #6 Solutions

1. $R = \frac{1}{m_1+m_2}(m_1 x_1 + m_2 x_2) \Rightarrow \dot{R} = \frac{1}{m_1+m_2}(m_1 v_1 + m_2 v_2)$

$r = x_1 - x_2 \Rightarrow \dot{r} = v_1 - v_2$, $E_{kin} = \frac{m_1+m_2}{2}\dot{R}^2 + \frac{\mu}{2}\dot{r}^2$, $\mu = \frac{m_1 m_2}{m_1+m_2}$

$$E_{kin} = \left(\frac{m_1+m_2}{2}\right)\left(\frac{1}{m_1+m_2}\right)^2(m_1 v_1 + m_2 v_2)^2 + \left(\frac{m_1 m_2}{m_1+m_2}\right)\left(\frac{1}{2}\right)(v_1 - v_2)^2$$

$$E_{kin} = \frac{1}{2}\left(\frac{1}{m_1+m_2}\right)(2m_1 m_2 v_1 v_2 + m_1^2 v_1^2 + m_2^2 v_2^2) + \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)(v_1^2 + v_2^2 - 2v_1 v_2)$$

$$E_{kin} = \frac{1}{2}\left(\frac{1}{m_1+m_2}\right)(2m_1 m_2 v_1 v_2 + m_1^2 v_1^2 + m_2^2 v_2^2 + m_1 m_2 v_1^2 + m_1 m_2 v_2^2 - 2m_1 m_2 v_1 v_2)$$

$$E_{kin} = \frac{1}{2}\left[\frac{v_1^2(m_1^2 + m_1 m_2)}{m_1+m_2} + \frac{v_2^2(m_2^2 + m_1 m_2)}{m_1+m_2}\right]$$

$$E_{kin} = \frac{1}{2}[m_1 v_1^2 + m_2 v_2^2] = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

2. Krane 4.2: $m = \mu \approx \frac{m_p}{2} \approx \frac{m_n}{2} = \frac{940 \text{ MeV}/c^2}{2}$, $E = -2.2$ MeV, $V_0 = 35$ MeV, $R = 2.1$ F

$u(r) = A\sin(k_1 r) + B\cos(k_1 r)$, $k_1 = (2m(E+V_0)/\hbar^2)^{1/2}$ $(r < R)$

Waveform must be finite $\Rightarrow B = 0 \Rightarrow u(r) = A\sin(k_1 r)$

$u(r) = C\exp(-k_2 r) + D\exp(k_2 r)$, $k_2 = (-2mE/\hbar^2)^{1/2}$ $(r > R)$

Waveform must be finite $\Rightarrow D = 0 \Rightarrow u(r) = C\exp(-k_2 r)$

$\psi(r) = \frac{u(r)}{r} \Rightarrow \psi_1(r) = \frac{A\sin(k_1 r)}{r}$ for $r < R$, $\psi_2(r) = \frac{C\exp(k_2 r)}{r}$ for $r > R$

Continuity at $r = R \Rightarrow \psi_1(r=R) = \psi_2(r=R) \Rightarrow C = A\sin(k_1 R)\exp(k_2 R)$

Normalization: $1 = |\psi|^2 = \int \psi^* \psi \, dr^3 = 4\pi\left[\int_0^R \frac{A^2 \sin^2(k_1 r)}{r^2} r^2 dr + \int_0^\infty \frac{C^2 \exp(-2k_2 r)}{r^2} r^2 dr\right]$

$$1 = 4\pi\left\{A^2\left[\frac{r}{2} - \frac{\sin(k_1 r)\cos(k_1 r)}{2k_1}\right]_0^R + C^2\left[-\frac{\exp(-2k_2 r)}{2k_2}\right]_R^\infty\right\}$$

$$1 = 4\pi\left[A^2\left(\frac{R}{2} - \frac{\sin(k_1 R)\cos(k_1 R)}{2k_1}\right) + C^2\left(\frac{\exp(-2k_2 R)}{2k_2}\right)\right]$$

$k_1 = 0.891$ F^{-1}, $k_2 = 0.231$ F^{-1} $\Rightarrow$

$$A = \left(\frac{1}{2\sqrt{\pi}}\right)\left[\frac{1}{\left(\frac{\pi}{2} - \frac{\sin(k_1 R)\cos(k_1 R)}{2k_1} + \frac{\sin^2(k_1 R)}{2k_2}\right)^{1/2}}\right] = \pm 0.158 \text{ F}^{-1/2} \Rightarrow \boxed{A = \pm 0.158 \text{ F}^{-1/2}}$$

$$C = \left(\frac{1}{2\sqrt{\pi}}\right)\left[\frac{\sin(k_1 R)\exp(k_2 R)}{\left(\frac{\pi}{2} - \frac{\sin(k_1 R)\cos(k_1 R)}{2k_1} + \frac{\sin^2(k_1 R)}{2k_2}\right)^{1/2}}\right] = \pm 0.245 \text{ F}^{-1/2} \Rightarrow \boxed{C = \pm 0.245 \text{ F}^{-1/2}}$$

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$r_{rms} = \sqrt{\langle r^2 \rangle}$, $\langle r^2 \rangle = \int \psi^* r^2 \psi \, dr^3 = 4\pi \left(A^2 \int_0^R r^2 \sin^2(k_1 r)\, dr + C^2 \int_R^\infty r^2 \exp(-2k_2 r)\, dr \right)$

$$\langle r^2 \rangle = 4\pi \left\{ A^2 \left[\frac{r^3}{6} - \frac{r\cos(2k_1 r)}{4k_1^2} - \frac{(-1+2k_1^2 r^2)\sin(2k_1 r)}{8k_1^3} \right]_0^R + C^2 \left[-\frac{(1+2k_2 r + 2k_2^2 r^2)\exp(-2k_2 r)}{4k_2^3} \right]_R^\infty \right\}$$

$\langle r^2 \rangle = 15.05 \, F^2 \Rightarrow \boxed{r_{rms} = 3.88 \, F}$

3. Krane 4.4: $P(r>R)\,dx = |\psi_2|^2 dx$

$$|\psi_2|^2 = 4\pi C^2 \int_R^\infty \frac{\exp(-2k_2 r)}{r^2} r^2 \, dr = 4\pi C^2 \frac{\exp(-2k_2 R)}{2k_2}, \quad C = \pm 0.245 \, F^{-1/2} \Rightarrow$$

$\boxed{P(r>R)\,dx = 0.620}$

4. Bound State Estimate: $\frac{\lambda}{4} \sim a$, First excited State: $\frac{3\lambda}{4} \sim a$

$k = \frac{2\pi}{\lambda}$, $E_{kin} = \frac{\hbar^2 k^2}{2m}$

Bound state: $k = \frac{\pi}{2R_0}$, $E_{kin} = \frac{(\hbar c)^2 k^2}{2Mc^2} = \frac{(197 \text{ MeV F})^2}{(940 \text{ MeV})} \left(\frac{\pi}{2(2.1 F)} \right)^2 = 23.1 \text{ MeV}$

Excited state: $\frac{3}{4}\lambda = R_0 \Rightarrow k = \frac{3\pi}{2R_0}$, $E_{kin} = 208$ MeV

208 MeV >> 23.1 MeV & > V_0 $\Rightarrow$ Unbound $\Rightarrow$ $\boxed{\text{No excited state from higher order eigenfunctions}}$

Non-zero angular momentum: $\ell \neq 0 \Rightarrow \ell = 1$

$$E_{kin-L} \simeq \frac{\hat{L}^2}{2\mu r^2} > \frac{\hbar^2 \ell(\ell+1)}{2\mu R_0^2} \simeq \frac{(\hbar c)^2 \ell(\ell+1)}{2\mu c^2 R_0^2} = \frac{(197 \text{ MeV F})^2}{(470 \text{ MeV})(2.1 F)^2} = 18.7 \text{ MeV} >> 2.2 \text{ MeV} \Rightarrow \text{Unbound}$$

$\boxed{\text{No excited state from non-zero angular momentum}}$

5. $V_{nuc}(r) = V_0(r) + V_1(r) \frac{\hat{s}_1 \cdot \hat{s}_2}{\hbar^2}$, $\hat{S} = \hat{s}_1 + \hat{s}_2$, $\hat{S}^2 = \hat{s}_1^2 + s_2^2 + 2\hat{s}_1 \cdot \hat{s}_2$

$\hat{s}_1 \cdot \hat{s}_2 = \frac{\hbar^2}{2} \left[S(S+1) - s_1(s_1+1) - s_2(s_2+1) \right]$

Pauli Exclusion Principle: Fermions must be antisymmetric $\Rightarrow S=0 \Rightarrow$ singlet state

$S=0$, $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2}$, $\hat{s}_1 \cdot \hat{s}_2 = \frac{\hbar^2}{2}\left(0 - \frac{3}{4} - \frac{3}{4}\right) = -\frac{3\hbar^2}{4}$

Deuteron case: $V_0 = -32.5$ MeV , $V_1 \simeq -10$ MeV

$V_{nuc}(r) = -32.5 \text{ MeV} + (-10 \text{ MeV})(-\frac{3}{4}) = -25 \text{ MeV} \Rightarrow$ unbound $\Rightarrow$

$\boxed{\text{Di-neutron cannot exist}}$