6.002 NOTES SUPPLEMENT: COMPLEX NUMBERS

A *complex number*, z for example, takes the form

$$z = a + jb \tag{1}$$

where a and b are both real numbers, and j is the *imaginary* unit defined according to

$$j^2 = -1. \tag{2}$$

Here, a is referred to as the *real part* of z, and b is referred to as the *imaginary part* of z. These two parts can be extracted from z using the real-part function $\Re()$ and the imaginary-part function $\Im()$, respectively. Thus, we write

$$a = \Re(z) \tag{3}$$

$$b = \Im(z), \tag{4}$$

and more generally,

$$z = \Re(z) + j\Im(z). \tag{5}$$

If $\Im(z) = 0$, then z is a purely real number. If $\Re(z) = 0$, then z is a purely imaginary number. Otherwise, z is a complex number.

A complex number can be viewed as a point in the two-dimensional *complex plane*, as shown in Figure 1 for z as given in Equation 1. The distance r measured from the origin to the point is referred to as the *magnitude* of z, and the angle θ measured from the real axis to the radius on which the point lies is referred to as the *angle*, or *phase*, of z. Thus we write

$$r = |z| \tag{6}$$

$$\theta = \angle z. \tag{7}$$

Similarly, a complex number can be viewed as a vector in the complex plane. In this case, its components are a and b, or $\Re(z)$ and $\Im(z)$, and its magnitude and direction are r and θ, or $|z|$ and $\angle z$.

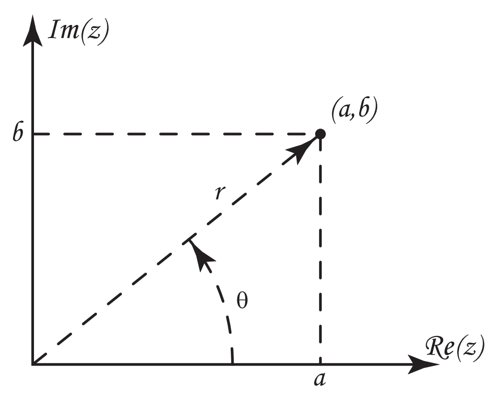

Figure 1: The location of the complex number z in the complex plane.

From the geometry of Figure 1 it is apparent that

$$r = \sqrt{a^2 + b^2} \tag{8}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right). \tag{9}$$

In Equation 9, the $\tan^{-1}()$ function is understood to be a two-argument inverse having full range such that $0 \leq \angle z < 2\pi$ or $-\pi \leq \angle z < \pi$; the choice of range is a matter of convenience. From Equations 8 and 9 we can identify the more general expressions

$$|z| = \sqrt{\Re(z)^2 + \Im(z)^2} \tag{10}$$

$$\angle z = \tan^{-1}\left(\frac{\Im(z)}{\Re(z)}\right). \tag{11}$$

Equations 10 and 11 express $|z|$ and $\angle z$ in terms of $\Re(z)$ and $\Im(z)$. It is also possible to invert these expressions. Again from the geometry of Figure 1, this yields

$$a = r\cos(\theta) \tag{12}$$

$$b = r\sin(\theta), \tag{13}$$

from which we may identify the general expressions

$$\Re(z) = |z|\cos(\angle z) \tag{14}$$

$$\Im(z) = |z|\sin(\angle z). \tag{15}$$

In summary, $\Re(z)$ and $\Im(z)$ are the Cartesian coordinates of z in the complex plane while $|z|$ and $\angle z$ are the polar coordinates of z in the same plane. Correspondingly, Equations 10 and 11 are a Cartesian-to-polar coordinate transformation, while Equations 14 and 15 are a polar-to-Cartesian coordinate transformation.

We can now use the polar coordinates $|z|$ and $\angle z$ to form an alternative expression for a complex number. Beginning with z as presented in Equation 1,

$$z = a + jb = r\cos(\theta) + jr\sin(\theta) = r(\cos(\theta) + j\sin(\theta)) = re^{j\theta}. \tag{16}$$

The first equality in Equation 16 follows from the substitution of Equations 12 and 13. The last equality follows from the substitution of the identity

$$\cos(\theta) + j\sin(\theta) \equiv e^{j\theta}, \tag{17}$$

which may be derived using Taylor Series expansions for the functions $e^{()}$, $\cos()$ and $\sin()$. From Equation 16 we can identify the more general expression

$$z = |z|e^{j\angle z}. \tag{18}$$

Equation 18 is the polar equivalent of Equation 5. We will use these two expressions interchangeably. Which one is preferred depends on the application. For example, we shall see shortly that addition and subtraction are most easily carried out using complex numbers in Cartesian form, while multiplication, division and determining a magnitude are most easily carried out using complex numbers in polar form.

Mathematical operations are performed on complex numbers just as they are on purely real numbers. For example, the addition and subtraction of the two complex numbers $(a_1 + jb_1)$ and $(a_2 + jb_2)$ proceeds according to

$$(a_1 + jb_1) + (a_2 + jb_2) = a_1 + jb_1 + a_2 + jb_2 = (a_1 + a_2) + j(b_1 + b_2) \tag{19}$$

$$(a_1 + jb_1) - (a_2 + jb_2) = a_1 + jb_1 - a_2 - jb_2 = (a_1 - a_2) + j(b_1 - b_2). \tag{20}$$

Thus, the real and imaginary parts of complex numbers add and subtract separately, just like the components of a vector. This is because the real and imaginary parts are defined along orthogonal axes of the complex plane. Because of this the addition and subtraction of complex numbers in polar form is less convenient.

The multiplication and division of complex numbers proceeds as directly as does their addition and subtraction. The only difference is that we commonly substitute -1 for each occurrence of j^2, as permitted by Equation 2. For example, the multiplication and division of the two complex numbers $(a_1 + jb_1)$ and $(a_2 + jb_2)$ proceeds according to

$$(a_1 + jb_1)(a_2 + jb_2) = a_1a_2 + ja_1b_2 + jb_1a_2 + j^2b_1b_2 = (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1) \tag{21}$$

$$\frac{a_1 + jb_1}{a_2 + jb_2} = \frac{a_1 + jb_1}{a_2 + jb_2} \cdot \frac{a_2 - jb_2}{a_2 - jb_2} = \frac{(a_1a_2 + b_1b_2) + j(b_1a_2 - a_1b_2)}{a_2^2 + b_2^2} = \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + j\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2}. \tag{22}$$

Note in particular the use of $(a_2 - jb_2)$ in Equation 22 to clear the denominator of terms involving j. While multiplication and division of complex numbers in Cartesian form is not inconvenient, it is much more convenient using complex numbers in polar form. For example,

$$r_1e^{j\theta_1}r_2e^{j\theta_2} = r_1r_2e^{j(\theta_1+\theta_2)} \tag{23}$$

$$\frac{r_1e^{j\theta_1}}{r_2e^{j\theta_2}} = \frac{r_1}{r_2}e^{j(\theta_1-\theta_2)}. \tag{24}$$

Equations 23 and 24 demonstrate that magnitudes multiply and divide, and that angles add and subtract, during the multiplication and division of complex numbers, respectively. Taking powers of complex numbers is equally convenient when the numbers are in polar form.

The *complex conjugate* of z, denoted here by z^* is defined such that

$$\Re(z^*) = \Re(z) \tag{25}$$

$$\Im(z^*) = -\Im(z). \tag{26}$$

For z as given in Equation 1,

$$z^* = a - jb, \tag{27}$$

while for z as given in Equation 16,

$$z^* = re^{-j\theta}. \tag{28}$$

In general, z^* can be derived from z by replacing each occurrence of j in z by $-j$.

By combining z with z^*, several useful relations can be expressed. In particular,

$$zz^* = |z|^2 \tag{29}$$

$$\frac{z + z^*}{2} = \Re(z) \tag{30}$$

$$\frac{z - z^*}{2j} = \Im(z). \tag{31}$$

$$\frac{1}{z} = \frac{z^*}{zz^*} = \frac{z^*}{|z|^2} \tag{32}$$

Each of these relations can be readily proven with the substitution of Equations 1 and 27. Equation 29 is a particularly useful means of computing $|z|$ when z is expressed in Cartesian form. In fact, it was used in Equation 22 to clear the denominator of terms involving j.

In Equations 16 and 17 we introduced the complex number $e^{j\theta}$. As we shall see shortly, numbers of this form are very important for our purposes, and to this end it has several important properties. First,

$$|e^{j\theta}| = 1 \tag{33}$$

$$\angle e^{j\theta} = \theta. \tag{34}$$

This can be seen by comparing $e^{j\theta}$ with Equation 17, or by substituting Equation 18 into Equation 29. Second,

$$\Re(e^{j\theta}) = \cos(\theta) \tag{35}$$

$$\Im(e^{j\theta}) = \sin(\theta) \tag{36}$$

These relations can be seen by comparing Equations 18 and 5. Finally, the multiplication of another complex number by $e^{j\theta}$ acts to rotate that number in the complex plane by the angle θ. To see this consider that

$$\left(r_1 e^{j\theta_1}\right)\left(e^{j\theta_2}\right) = r_1 e^{j(\theta_1+\theta_2)}. \tag{37}$$

Here, the multiplication of $r_1 e^{j\theta_1}$ by $e^{j\theta_2}$ preserves the magnitude r_1 while adding θ_2 to the angle θ_1. Thus, the complex number $r_1 e^{j\theta_1}$ is rotated in the complex plane by the angle θ_2.

Our discussion of complex numbers to this point has focused on constant complex numbers. However, none of the discussion has actually relied on z being constant. Indeed, the entire discussion applies to complex functions of time, and in particular to the time function $e^{j\omega t}$. This time function is central to our study of the sinusoidal steady state operation of linear electronic circuits. Following Equations 33 through 36, we see that

$$|e^{j\omega t}| = 1 \tag{38}$$

$$\angle e^{j\omega t} = \omega t \tag{39}$$

$$\Re(e^{j\omega t}) = \cos(\omega t) \tag{40}$$

$$\Im(e^{j\omega t}) = \sin(\omega t). \tag{41}$$

Equations 38 and 39 show that $e^{j\omega t}$ is a unit vector in the complex plane that rotates with the angular frequency ω. Equations 40 and 41 are projections of this vector onto the real and imaginary axes, respectively.

We close this appendix with a several numerical examples. For these examples, let

$$z_1 = -2 + j2 \tag{42}$$

$$z_2 = 1 + j\sqrt{3}. \tag{43}$$

We can find the complex conjugates of z_1 and z_2 using Equations 25 and 26. This yields

$$z_1^* = -2 - j2 \tag{44}$$

$$z_2^* = 1 - j\sqrt{3}. \tag{45}$$

We can find the real and imaginary parts of z_1 and z_2 by association with Equation 5, or by using Equations 30 and 31. By either method,

$$\Re(z_1) = -2; \quad \Im(z_1) = 2 \tag{46}$$

$$\Re(z_2) = 1; \quad \Im(z_2) = \sqrt{3}. \tag{47}$$

We can find the magnitude and angle of z_1 and z_2 using Equations 10 and 11. This yields

$$|z_1| = 2\sqrt{2}; \quad \angle z_1 = \frac{3\pi}{4} \tag{48}$$

$$|z_2| = 2; \quad \angle z_2 = \frac{\pi}{3}. \tag{49}$$

We can find the polar form of z_1 and z_2 using Equation 18. This yields

$$z_1 = 2\sqrt{2}e^{j\frac{3\pi}{4}} \tag{50}$$

$$z_2 = 2e^{j\frac{\pi}{3}}. \tag{51}$$

Following Equations 19 and 20, the sum and difference of z_1 and z_2 are

$$z_1 + z_2 = -1 + j\left(2 + \sqrt{3}\right); \quad z_1 - z_2 = -3 + j\left(2 - \sqrt{3}\right). \tag{52}$$

Following Equations 21 and 22, the product and ratio of z_1 and z_2 are

$$z_1 z_2 = -\left(2\sqrt{3} + 2\right) - j\left(2\sqrt{3} - 2\right); \quad \frac{z_1}{z_2} = \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) + j\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right). \tag{53}$$

Alternatively, following Equations 23 and 24, the product and ratio of z_1 and z_2 are

$$z_1 z_2 = 4\sqrt{2}e^{j\frac{13\pi}{12}}; \quad \frac{z_1}{z_2} = \sqrt{2}e^{j\frac{5\pi}{12}}. \tag{54}$$