MASSACHVSETTS INSTITVTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.002 – Electronic Circuits
Fall 2001

Problem Set 11 Solutions

Solutions:

1. (A) As shown in Figure 1, the capacitor now begins to discharge from V_H toward zero. The capacitor voltage is given by

$$v_{CAP} = V_H e^{-t/RC}.$$

We want to solve for the elapsed time from $v_{CAP} = V_H$ to $v_{CAP} = V_L$. This time is denoted by T_2 in Figure 1. Thus,

$$V_L = V_H e^{-T_2/RC}.$$

We solve:

$$T_2 = RC \log \frac{V_H}{V_L}.$$

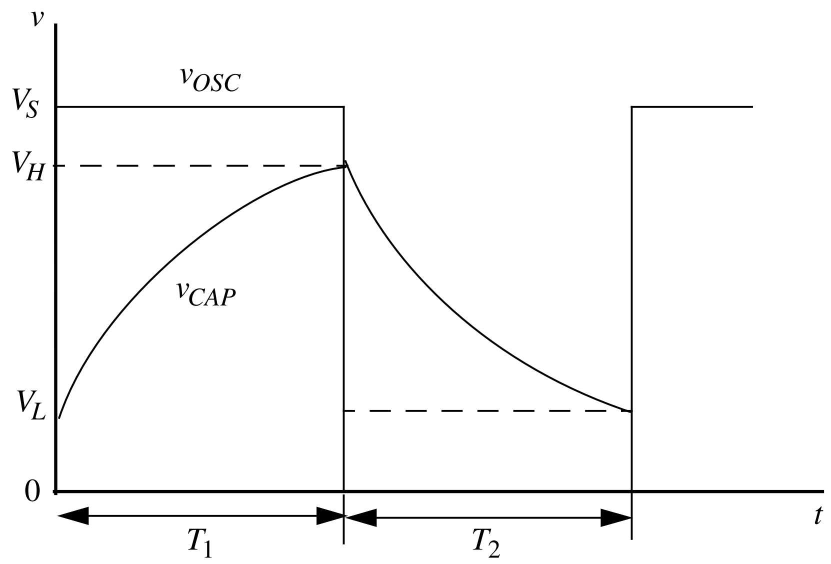

Figure 1: v_{CAP} and v_{OSC}.

(B) Now the capacitor starts to charge from V_L toward V_S. The capacitor voltage is

$$v_{CAP} = V_S + (V_L - V_S)e^{-t/RC}.$$

We want to solve for the elapsed time from $v_{CAP} = V_L$ to $v_{CAP} = V_H$. This time is denoted by T_1 in Figure 1. Thus,

$$V_H = V_S + (V_L - V_S)e^{-T_1/RC}.$$

We solve:

$$T_1 = RC \log \frac{V_S - V_L}{V_S - V_H}.$$

(C) The period T of the oscillator is

$$T = T_1 + T_2 = RC \log \frac{(V_S - V_L)V_H}{(V_S - V_H)V_L}.$$

The frequency f is the inverse of the period and is given by

$$f = \frac{1}{T} = \frac{1}{RC \log \frac{(V_S - V_L)V_H}{(V_S - V_H)V_L}}.$$

(D) $f = 8$ kHz is equivalent to $T = 1/f = 125\ \mu$s. We choose $C = 0.0047\ \mu$F, and solve for R using the equation in Part (C):

$$R = \frac{T}{C \log \frac{(V_S - V_L)V_H}{(V_S - V_H)V_L}} = 27\ \text{k}\Omega,$$

which is a standard resistor value.

(E) Using the equations developed in (A) and (B) we get $T_1 = 59.6\ \mu$s and $T_2 = 64.8\ \mu$s. We notice that $T_1 \neq T_2$. This does not affect the operation of the clock because the counter is triggered by either the rising or the falling edge, but not both.

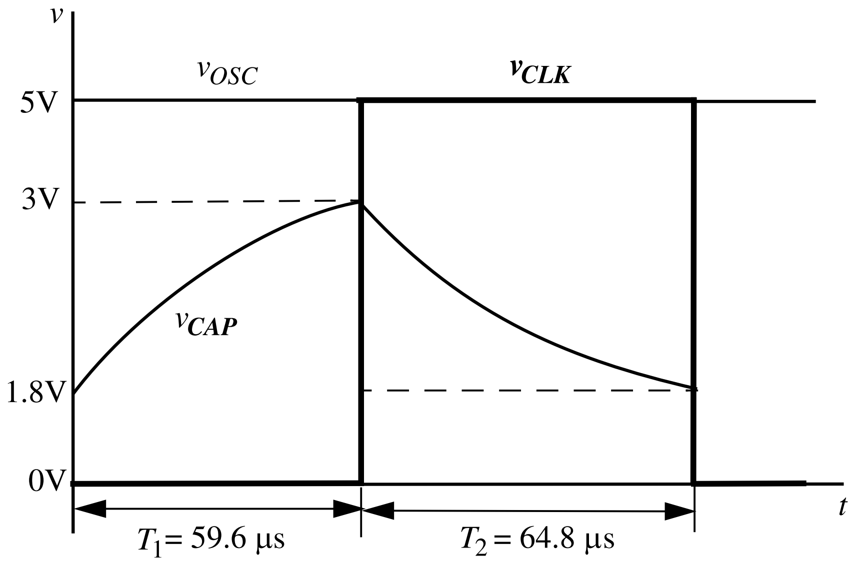

Figure 2: v_{CAP}, v_{OSC}, and v_{CLK}.

2. (A) The resistive network at the negative terminal of the op-amp is called an R-2R ladder. We can solve by using the superposition principle. We turn off all the sources except one, say v_{DB0}, and then apply Thevenin's theorem repeatedly across each "T" node from left to right, as shown in Figure 3.

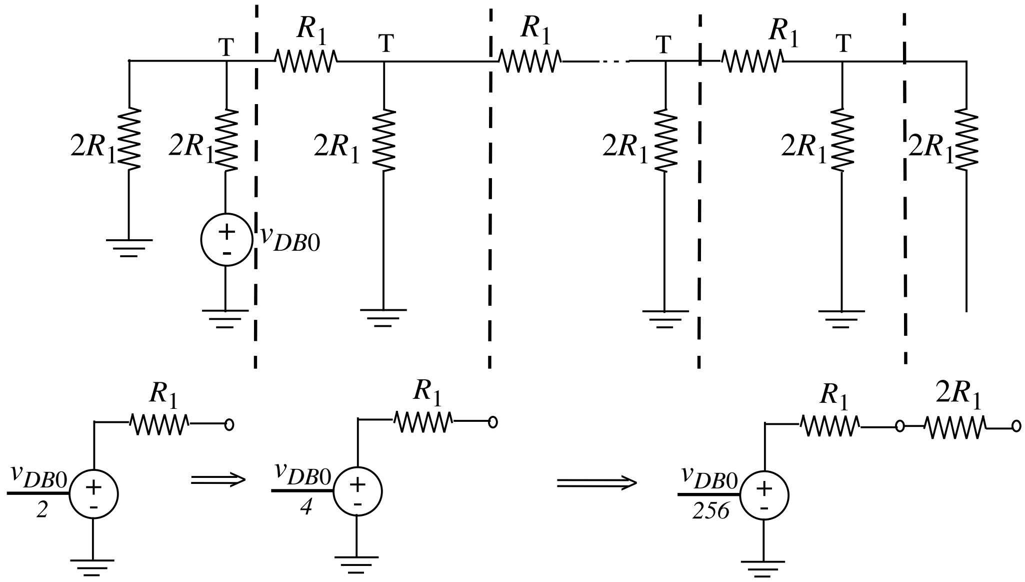

Figure 3: Thevenin equivalent process.

As demonstrated above, the Thevenin voltage is cut by half across each "T" node, and the Thevenin resistance is always R_1. Eventually we simplify the network into one source in series with one resistor at the negative terminal of the op-amp. Since we have turned v_{OFF} off, what we have now is a simple inverting configuration, and the output is given by

$$v_{DAC} = -\frac{R_2}{3R_1}\frac{v_{DB0}}{2^8}.$$

Repeating the same procedure for each v_{DBi}, with v_{OFF} set to zero, we arrive at the following equation:

$$\begin{aligned} v_{DAC} &= -\frac{R_2}{3R_1}\sum_{i=0}^{7}\frac{v_{DBi}}{2^{8-i}}, \\ &= -\frac{R_2}{3R_1}\sum_{i=0}^{7}\frac{2^i}{2^8}v_{DBi}. \end{aligned}$$

Next we consider the effect of v_{OFF}, when all the v_{DBi}'s are set to zero. The resistance R_- looking into the R-2R ladder is $R_- = R_1 + 2R_1 = 3R_1$, as shown in Figure 3. It then follows that

$$v_{DAC} = v_{OFF}\left(1 + \frac{R_2}{3R_1}\right).$$

Finally, the total solution is

$$v_{DAC} = -\frac{R_2}{3R_1}\sum_{i=0}^{7}\frac{2^i}{2^8}v_{DBi} + v_{OFF}\left(1+\frac{R_2}{3R_1}\right).$$

(B) The v_{DAC} equation provided must equal the one derived in Part (A). When all the bits are on,

$$v_{DAC} = -2.5 \text{ V} = -\frac{R_2}{3R_1}\frac{255}{256}5 \text{ V}.$$

Solving for R_2, we get

$$R_2 = \frac{3\cdot 256}{510}R_1 = 1.51R_1.$$

(C) Setting all bits to 0, we get

$$v_{DAC} = 1.25 = v_{OFF}\left(1+\frac{R_2}{3R_1}\right) = v_{OFF}\left(1+\frac{256}{510}\right).$$

This gives us

$$v_{OFF} = 0.83 \text{ V}.$$

(D) R_2 is found by directly applying the equation found in Part (B):

$$R_2 = 1.51R_1 = 15.1 \text{ k}\Omega.$$

3. (A) Unfortunately, there is no simple way to solve this circuit. We are forced to return to the Node Method. There are three nodes in the circuit, but the Ideal Op-Amp Model allows us to assume that $v_+ = v_- = v_{LPF}$. By using this simplification, we can avoid writing a node equation for the output node. We define the potential of the node connecting the resistors as E.

For the v_+ node:

$$\frac{V_{lpf}}{\frac{1}{j\omega C_2}} + \frac{V_{lpf}-E}{R} = 0.$$

For the E node:

$$\frac{V_{lpf}-E}{\frac{1}{j\omega C_1}} + \frac{V_{lpf}-E}{R} + \frac{V_{dac}-E}{R} = 0.$$

We choose to solve the first equation for E and substitute into the second. Mathematically, it is irrelevant how we solve, but practically, it tends to be easier to solve the simplest equation first, and substitute into more complicated expressions. Solving the v_+ node equation gives us

$$E = V_{lpf}(1+j\omega RC_2).$$

Substituting in for E and factoring out a V_{lpf} gives us

$$V_{lpf}\left[\frac{(1+j\omega RC_2)-1}{\frac{1}{j\omega C_1}} + \frac{(1+j\omega RC_2)-1}{R} + \frac{1+j\omega RC_2}{R}\right] = \frac{V_{dac}}{R}.$$

Solving for $\frac{V_{lpf}}{V_{dac}}$, we have

$$H_{LPF}(\omega) = \frac{V_{lpf}}{V_{dac}} = \frac{1}{1 - \omega^2 R^2 C_1 C_2 + 2j\omega RC_2}.$$

(B) From Part (A):

$$|H_{LPF}(\omega)| = \frac{1}{\sqrt{(1 - \omega^2 R^2 C_1 C_2)^2 + (2\omega RC_2)^2}}.$$

$$\angle H_{LPF}(\omega) = 0 - \arctan \frac{2\omega RC_2}{1 - \omega^2 R^2 C_1 C_2}.$$

(C) The low- and high-frequency asymptotes are calculated by including only the lowest- and highest-order ω terms, respectively. We simplify in this manner until we are left with a single term in the denominator.

For $\omega \ll \omega_{LPF}$ we drop the ω term because it is much less than 1. The result is:

$$H_{LPF}(\omega \ll \omega_{LPF}) \approx 1.$$

For $\omega \gg \omega_{LPF}$ we drop the 1 term because it is much less than ω. The result is:

$$H_{LPF}(\omega \gg \omega_{LPF}) \approx \frac{1}{(\omega/\omega_{LPF})^2}.$$

Solving for the intersection of these equations gives the expected $\omega = \omega_{LPF}$. This is known as the cut-off frequency.

(D) The current equation for $|H_{LPF}(\omega)|$ does not look much like our desired result, so we try multiplying out the denominator:

$$|H_{LPF}(\omega)| = \frac{1}{\sqrt{1 - 2\omega^2 R^2 C_1 C_2 + (\omega^2 R^2 C_1 C_2)^2 + 4\omega^2 R^2 C_2^2}}.$$

If we let $C_1 = C_2 = C$, then we can easily factor that equation:

$$|H_{LPF}(\omega)| = \frac{1}{\sqrt{1 + 2\omega^2 R^2 C^2 + (\omega^2 R^2 C^2)^2}} = \frac{1}{\sqrt{(1 + \omega^2 R^2 C^2)^2}} = \frac{1}{1 + \omega^2 R^2 C^2}.$$

This final form looks a lot more like what we want. We can see that $\omega_{LPF} = 1/(RC)$.

(E) We see that $RC = 1/(2\pi \times 4000) \approx 3.98 \times 10^{-5}$. To choose a good value for R and C we need to consider the parasitic elements in the circuit.

There are parasitic capacitances between the holes in the protoboard. Those tend to be on the order of single picofarads. We need our C value to be much larger than that to be unaffected by these parasitic capacitances.

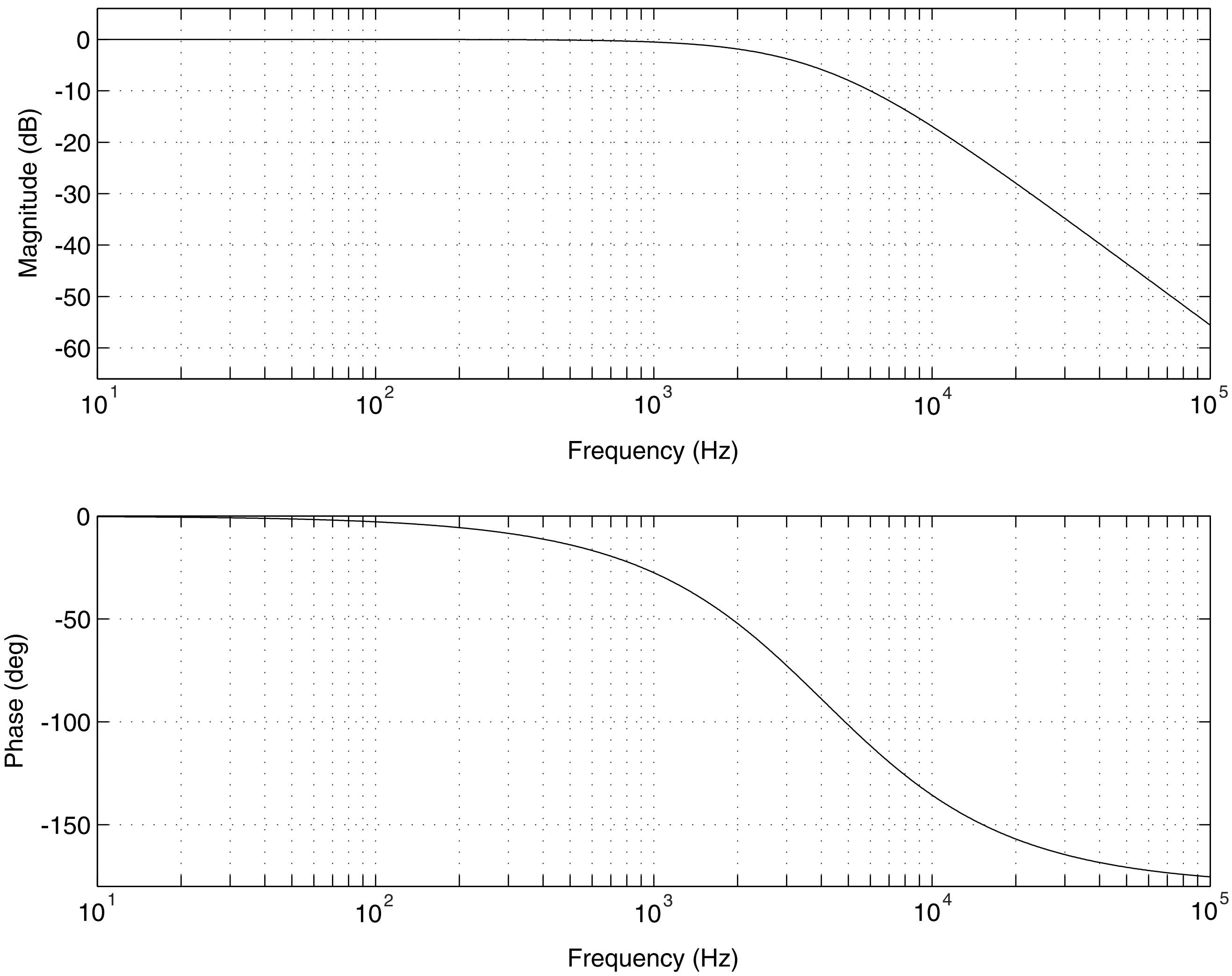

Figure 4: Frequency response curves.

The resistor choice is affected by the presence of source resistances and input and output resistances of the op-amps. In addition there are parasitic inductances. If the resistance is too small, the parasitic RLC circuits may be underdamped and experience ringing. In general we choose a resistance of a few kΩ. We have an $R = 3.9$ kΩ in the lab kit, so we will use that.

$R = 3.9$ kΩ requires that $C = 0.0102$ μF. Since this is much larger than a picofarad, and we have a 0.01 μF capacitor in our kit, it seems that these values are a good choice.

Just to check, we see that our $RC = 3.9 \times 10^{-5}$. This is about 2% different than the desired time constant, which is much less than the resistor or the capacitor tolerances.

(F) From the above we see that $\omega_{LPF} = 2.56 \times 10^4 = 2\pi \times 4081$ rad/s. The plots are in Figure 4.