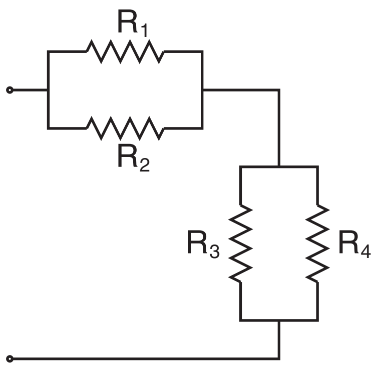
MASSACHVSETTS INSTITVTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

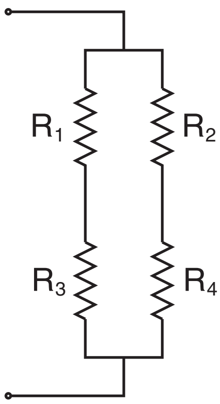
6.002 – Electronic Circuits
Fall 2001

Problem Set 2 Solutions

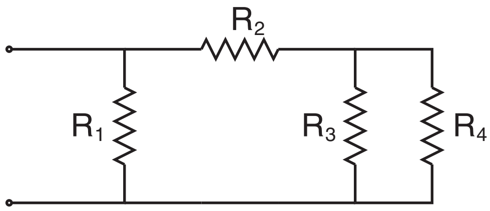
Problem 2-1. Calculate the Thévenin equivalent resistance at the indicated port for each of the networks shown below.



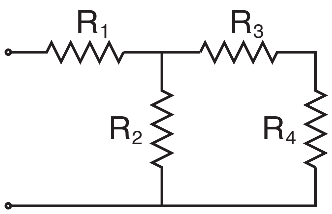
(i)



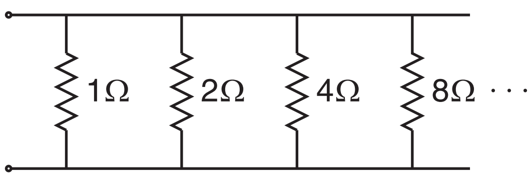
(ii)



(iii)



(iv)



(v)

$$(i)\quad R_{Th} = R_1||R_2 + R_3||R_4 = \frac{R_1R_2}{R_1+R_2} + \frac{R_3R_4}{R_3+R_4} = \frac{R_1R_2R_3 + R_1R_2R_4 + R_1R_3R_4 + R_2R_3R_4}{R_1R_3 + R_1R_4 + R_2R_3 + R_2R_4}$$

$$(ii)\quad R_{Th} = (R_1+R_3)||(R_2+R_4) = \frac{(R_1+R_3)(R_2+R_4)}{R_1+R_2+R_3+R_4} = \frac{R_1R_2 + R_1R_4 + R_2R_3 + R_3R_4}{R_1+R_2+R_3+R_4}$$

$$(iii)\quad R_{Th} = R_1||(R_2 + R_3||R_4) = R_1||\frac{R_2R_3 + R_2R_4 + R_3R_4}{R_3+R_4}$$

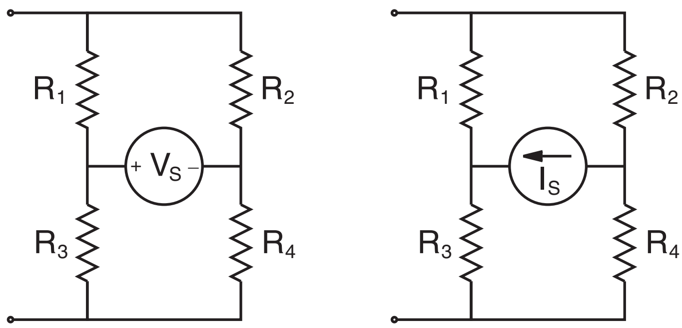
$$= \frac{R_1R_2R_3 + R_1R_2R_4 + R_1R_3R_4}{R_1R_3 + R_1R_4 + R_2R_3 + R_2R_4 + R_3R_4}$$

$$(iv)\quad R_{Th} = R_1 + R_2||(R_3+R_4) = R_1 + \frac{R_2R_3 + R_2R_4}{R_2+R_3+R_4} = \frac{R_1R_2 + R_1R_3 + R_1R_4 + R_2R_3 + R_2R_4}{R_2+R_3+R_4}$$

$$(v)\quad G_{Th} = 1S + 1/2S + 1/4S + 1/8S + \ldots = 2S \Rightarrow R_{Th} = 1/2\Omega$$

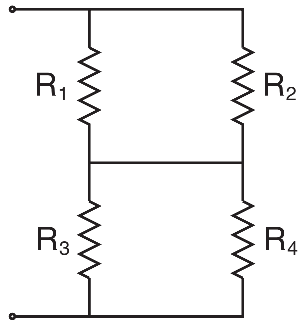
Note: The S stands for Siemens, the SI units for conductance ($1S = 1\Omega^{-1}$). An equivalent unit is the mhos, whose symbol is an Ω flipped upside down, but since I can't figure out how to write it in LaTeX, I'll stick to Siemens (S).

Problem 2-2. Find both Thévenin and Norton equivalent circuits at the indicated ports for the two networks shown below. Note that the results from Problem 2-1 may be useful in calculating R_T, while voltage and current divider relations are useful in finding V_T or I_N.



(i) (ii)

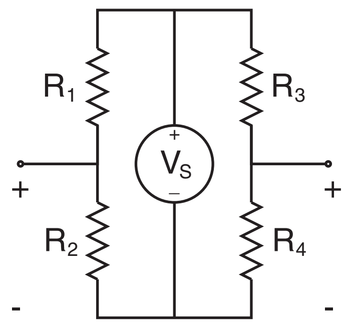
(i) To find R_{Th}, we first set the voltage source to zero.



This circuit is identical to 2-1(i). Thus,

$$R_{Th} = R_1||R_2 + R_3||R_4 = \frac{R_1R_2R_3 + R_1R_2R_4 + R_1R_3R_4 + R_2R_3R_4}{R_1R_3 + R_1R_4 + R_2R_3 + R_2R_4}$$

One way to find the open circuit voltage, V_{OC}, is to redraw the figure as follows.



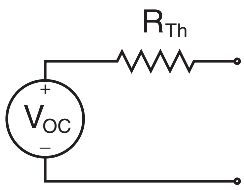
Thus,

$$V_{R_2} = \frac{R_2}{R_1 + R_2}V_S \quad \text{and} \quad V_{R_4} = \frac{R_4}{R_3 + R_4}V_S$$

Therefore,

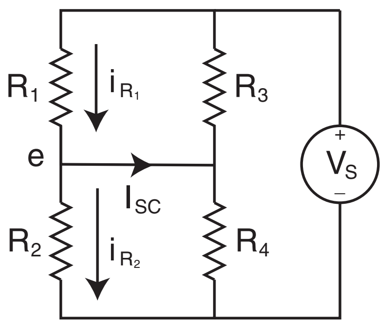
$$V_{OC} = V_{R_2} - V_{R_4} = \left(\frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4}\right)V_S = \frac{R_2R_3 - R_1R_4}{R_1R_3 + R_1R_4 + R_2R_3 + R_2R_4}V_S$$

The resulting Thevénin equivalent circuit is



with $V_{OC} = \frac{R_2R_3 - R_1R_4}{R_1R_3+R_1R_4+R_2R_3+R_2R_4}V_S$ and $R_{Th} = \frac{R_1R_2R_3+R_1R_2R_4+R_1R_3R_4+R_2R_3R_4}{R_1R_3+R_1R_4+R_2R_3+R_2R_4}$.

To find the Norton equivalent, compute the short circuit current by shorting the terminals,



The voltage e can be found from a voltage divider relationship.

$$e = \frac{R_2||R_4}{R_2||R_4 + R_1||R_3}V_S = \frac{R_1R_2R_4 + R_2R_3R_4}{R_1R_2R_3 + R_1R_2R_4 + R_1R_3R_4 + R_2R_3R_4}V_S$$

Given that voltage, we can compute I_{SC} as the difference between i_{R_1} and i_{R_2}.

$$I_{SC} = i_{R_1} - i_{R_2} = \frac{V - e}{R_1} - \frac{e}{R_2}$$

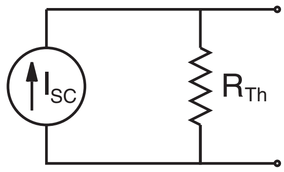
After substituting the equation for the voltage e into the previous expression and cranking the algebra, we obtain

$$I_{SC} = \frac{R_2R_3 - R_1R_4}{R_1R_2R_3 + R_1R_2R_4 + R_1R_3R_4 + R_2R_3R_4}V_S$$

Another (presumably easier method, given that you have V_{OC} and R_{Th}) is to rely on the relationship between the Thévenin Resistance, open circuit voltage and short circuit current.

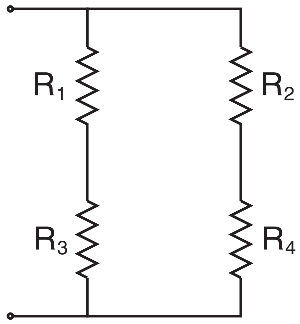
$$I_{SC} = \frac{V_{OC}}{R_{Th}} = \frac{R_2R_3 - R_1R_4}{R_1R_2R_3 + R_1R_2R_4 + R_1R_3R_4 + R_2R_3R_4}V_S$$

The resulting Norton equivalent circuit is



with $I_{SC} = \frac{R_2R_3 - R_1R_4}{R_1R_2R_3 + R_1R_2R_4 + R_1R_3R_4 + R_2R_3R_4} V_S$ and $R_{Th} = \frac{R_1R_2R_3 + R_1R_2R_4 + R_1R_3R_4 + R_2R_3R_4}{R_1R_3 + R_1R_4 + R_2R_3 + R_2R_4}$.

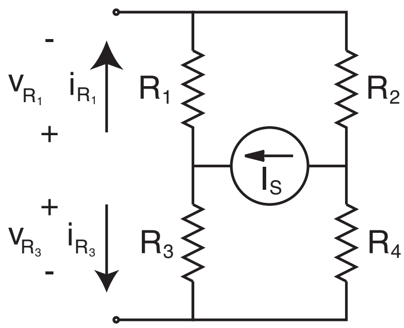
(ii) To find the Thévenin resistance, R_{Th}, we first turn off the current source.



This circuit is identical to 2-1(ii). Thus, the resistance is

$$R_{Th} = (R_1 + R_3)||(R_2 + R_4) = \frac{R_1R_2 + R_1R_4 + R_2R_3 + R_3R_4}{R_1 + R_2 + R_3 + R_4}$$

To find the open circuit voltage, we note that the current from the source branches off into R_1 and R_3.



We can compute both currents using a current divider:

$$i_{R_1} = \frac{R_3 + R_4}{R_1 + R_2 + R_3 + R_4} I_S \quad \text{and} \quad i_{R_3} = \frac{R_1 + R_2}{R_1 + R_2 + R_3 + R_4} I_S$$

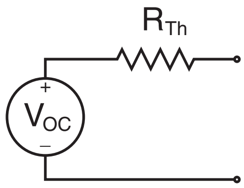
These two currents generate voltages across R_1 and R_3 equal to:

$$v_{R_1} = \frac{R_1R_3 + R_1R_4}{R_1 + R_2 + R_3 + R_4} I_S \quad \text{and} \quad v_{R_3} = \frac{R_1R_3 + R_2R_3}{R_1 + R_2 + R_3 + R_4} I_S$$

Thus,

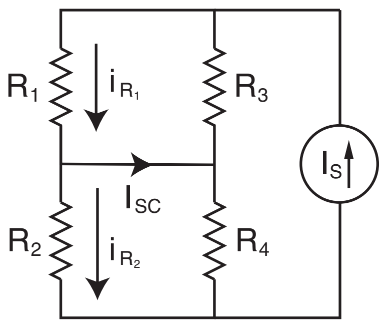
$$v_{OC} = v_{R_3} - v_{R_1} = \frac{R_2R_3 - R_1R_4}{R_1 + R_2 + R_3 + R_4} I_S$$

The resulting Thevénin equivalent circuit is



with $V_{OC} = \frac{R_2R_3 - R_1R_4}{R_1+R_2+R_3+R_4} I_S$ and $R_{Th} = \frac{R_1R_2+R_1R_4+R_2R_3+R_3R_4}{R_1+R_2+R_3+R_4}$.

To find the Norton equivalent, we first redraw the circuit as follows



Currents i_{R_1} and i_{R_2} can be computed using current divider relationships,

$$i_{R_1} = \frac{R_3}{R_1 + R_3} I_S \quad \text{and} \quad -i_{R_2} = -\frac{R_2}{R_2 + R_4} I_S$$

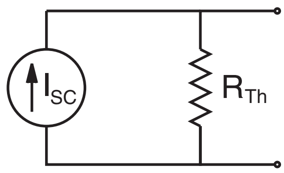
Again, I_{SC} is simply the difference between these two currents

$$I_{SC} = i_{R_1} - i_{R_2} = \left(\frac{R_3}{R_1 + R_3} - \frac{R_2}{R_2 + R_4} \right) I_S = \frac{R_2R_3 - R_1R_4}{R_1R_2 + R_1R_4 + R_2R_3 + R_3R_4} I_S$$

Alternatively, or as a check, we again can rely on the relationship between the Thévenin Resistance, open circuit voltage and short circuit current.

$$I_{SC} = \frac{V_{OC}}{R_{Th}} = \frac{R_2R_3 - R_1R_4}{R_1R_2 + R_1R_4 + R_2R_3 + R_3R_4} I_S$$

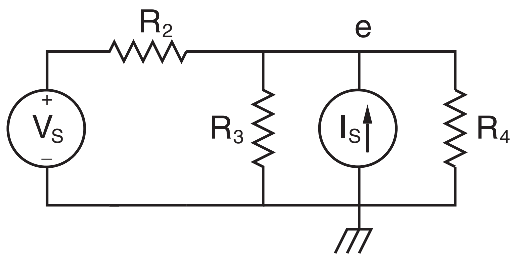
The resulting Norton equivalent circuit is



with $I_{SC} = \frac{R_2R_3 - R_1R_4}{R_1R_2+R_1R_4+R_2R_3+R_3R_4} I_S$ and $R_{Th} = \frac{R_1R_2+R_1R_4+R_2R_3+R_3R_4}{R_1+R_2+R_3+R_4}$.

Note that in both circuits, if $V_S = I_S$ and $R_1 = R_2 = R_3 = R_4 = R$, the resulting Thévenin or Norton equivalent is simply a resistor of value R. This is a useful property of this network configuration called a *bridge circuit*. Any source in the interior of the bridge "disappears" at the terminals, as long as $R_2R_3 = R_1R_4$, a condition referred to as a *balanced bridge*. Thus, this circuit is primarily used to isolate sources from another parts of a circuit, or to measure the resistance of part of the bridge (i.e., making R_1 a function of temperature, stress, or pH).

Problem 2-3. Refer to the network shown below for parts (a) and (b).



(a) Write and solve a node equation determining the node voltage e.

KCL at the node with voltage e implies

$$\frac{V_S - e}{R_2} + I_S = \frac{e-0}{R_3} + \frac{e-0}{R_4}$$

We can rewrite the equation in the standard form as

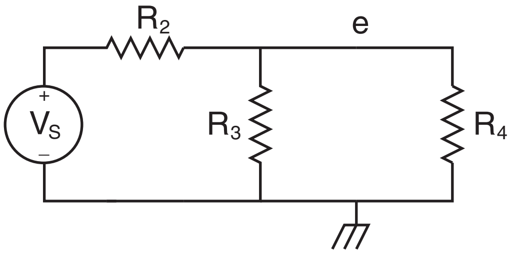
$$\left(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) e = \frac{V_S}{R_2} + I_S$$

Solving for e, we find

$$e = \frac{R_3R_4}{R_2R_3 + R_2R_4 + R_3R_4} V_S + \frac{R_2R_3R_4}{R_2R_3 + R_2R_4 + R_3R_4} I_S$$

(b) Find the node voltage e using superposition, i.e., determine e due to each source acting alone and adding the results. Compare with the answer found in part (a).

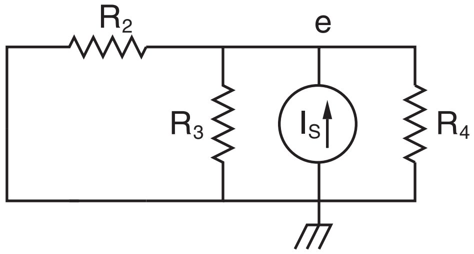
First, turn off the current source



We can quickly observe that

$$e|_{I_S=0} = \frac{R_3||R_4}{R_2 + R_3||R_4} V_S = \frac{R_3 R_4}{R_2 R_3 + R_2 R_4 + R_3 R_4} V_S$$

Now we turn off the voltage source



Again, we quickly deduce that

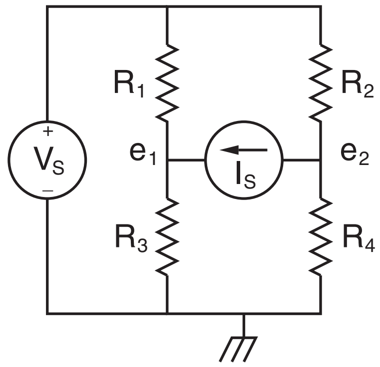
$$e|_{V_S=0} = R_2||R_3||R_4 I_S = \frac{R_2 R_3 R_4}{R_2 R_3 + R_2 R_4 + R_3 R_4} I_S$$

Thus, the total voltage at the node is

$$e = e|_{I_S=0} + e|_{V_S=0} = \frac{R_3 R_4}{R_2 R_3 + R_2 R_4 + R_3 R_4} V_S + \frac{R_2 R_3 R_4}{R_2 R_3 + R_2 R_4 + R_3 R_4} I_S$$

The answer is identical to part (a), as expected.

Refer to the network shown below for parts (c) and (d).



(c) Write and solve node equations determining the two node voltages e_1 and e_2.

KCL at node e_1, assuming all currents flowing *into* the node yields

$$\frac{V_S - e_1}{R_1} + I_S + \frac{0 - e_1}{R_3} = 0$$

A similar analysis at node e_2 results in

$$\frac{V_S - e_2}{R_2} - I_S + \frac{0 - e_2}{R_4} = 0$$

Replacing resistances for conductances and casting the two equation in matrix form produces

$$\begin{bmatrix} G_1 + G_3 & 0 \\ 0 & G_2 + G_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} I_S + V_S G_1 \\ -I_S + V_S G_2 \end{bmatrix}$$

Inverting the matrix of conductances,

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{G_1+G_3} & 0 \\ 0 & \frac{1}{G_2+G_4} \end{bmatrix} \begin{bmatrix} I_S + V_S G_1 \\ -I_S + V_S G_2 \end{bmatrix}$$

Thus,

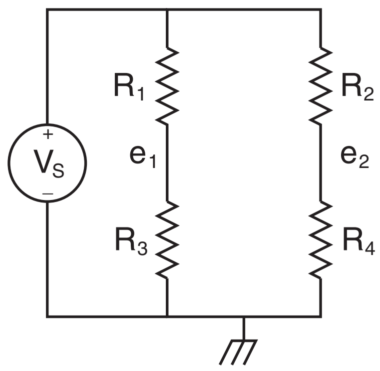
$$e_1 = \frac{I_S + V_S G_1}{G_1 + G_3} \quad \text{and} \quad e_2 = \frac{-I_S + V_S G_2}{G_2 + G_4}$$

Replacing the conductances for resistances, we obtain

$$e_1 = \frac{I_S R_1 R_3 + V_S R_3}{R_1 + R_3} \quad \text{and} \quad e_2 = \frac{-I_S R_2 R_4 + V_S R_4}{R_2 + R_4}$$

(d) Solve for e_1 and e_2 using superposition and compare your results with those from part (c).

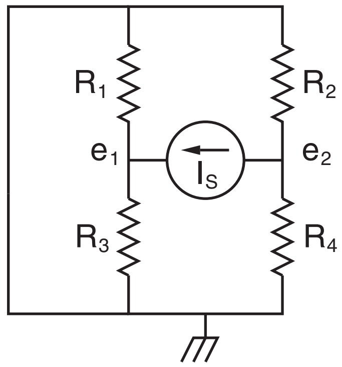
Turning off the current source, we are left with



Using voltage divider relations, we get

$$e_1|_{I_S=0} = \frac{R_3}{R_1+R_3}V_S \quad \text{and} \quad e_2|_{I_S=0} = \frac{R_4}{R_2+R_4}V_S$$

We then turn off the voltage source



$$e_1|_{V_S=0} = R_1||R_3 I_S = \frac{R_1R_3}{R_1+R_3}I_S \quad \text{and} \quad e_2|_{V_S=0} = -R_2||R_4 I_S = \frac{R_2R_4}{R_2+R_4}I_S$$

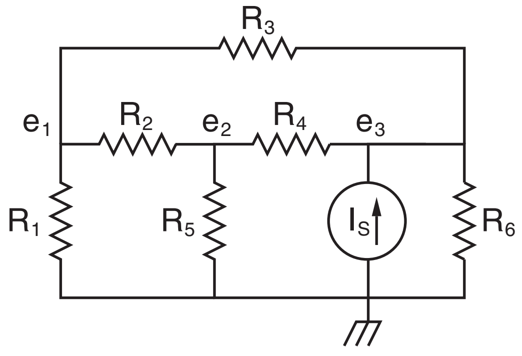
Combining these results, we obtain

$$e_1 = e_1|_{I_S=0} + e_1|_{V_S=0} = \frac{R_3}{R_1+R_3}V_S + \frac{R_1R_3}{R_1+R_3}I_S$$

$$e_2 = e_2|_{I_S=0} + e_2|_{V_S=0} = \frac{R_4}{R_2+R_4}V_S + \frac{R_2R_4}{R_2+R_4}I_S$$

as expected.

Problem 2-4. Refer to the network shown below.



(a) Write a set of node equations for the node-to-datum voltages e_1, e_2 and e_3 in the above network. Put the equations in matrix form:

$$G \cdot [e] = [s]$$

where G is a 3X3 matrix of conductance terms, e is a vector containing the three node voltages e_1, e_2 and e_3, and s is a vector containing source values.

KCL at the three labeled nodes yields the following set of equations (note that I will write them using conductances).

$$(e_3 - e_1)G_3 + (0 - e_1)G_1 + (e_2 - e_1)G_2 = 0$$

$$(e_1 - e_2)G_2 + (e_3 - e_2)G_4 + (0 - e_2)G_5 = 0$$

$$(e_1 - e_3)G_3 + (e_2 - e_3)G_4 + (0 - e_3)G_6 + I_S = 0$$

Casting these equations in matrix form results in

$$\begin{bmatrix} G_1 + G_2 + G_3 & -G_2 & -G_3 \\ -G_2 & G_2 + G_4 + G_5 & -G_4 \\ -G_3 & -G_4 & G_3 + G_4 + G_6 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_S \end{bmatrix}$$

(b) Solve these equations for the special case $G_1 = G_2 = G_3 \ldots = G_6 = 1mho$.

Substituting the given values into the matrix, we obtain

$$\begin{bmatrix} 3S & -1S & -1S \\ -1S & 3S & -1S \\ -1S & -1S & 3S \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_S \end{bmatrix}$$

Solving this matrix equation,

$$e_1 = \frac{I_S}{4S} \quad e_2 = \frac{I_S}{4S} \quad e_3 = \frac{I_S}{2S}$$

(c) For the values of the G's given in (b), what is the current in R_2?

Since $e_1 = e_2$, there is no voltage across R_2, and by Ohm's law, we deduce that $i_{R_2} = 0$. Note that this is again the bridge circuit bent out of shape. By now, you are experts in it!