6.002 Fall 2001 Problem Set 7 Solutions

Prologue: A long, long time ago, the staff handed out a calibration quiz. On it was a simple question: Solve the differential equation

$$\frac{dx}{dt}+\alpha x=\beta$$

This can be solved uniquely if you know one boundary condition. The solution proceeds by finding the homogeneous solution, a particular solution, adding the two, and then matching the boundary condition. The homogeneous form of the differential equation is

$$\frac{dx}{dt}+\alpha x = 0$$

Assume a solution of the form:

$$x(t)=c_1 e^{At}$$

Then:

$$\frac{dx}{dt} = A c_1 e^{At} = A x$$

Since $A x+\alpha x = 0$ we find that $A = -\alpha$. Our homogeneous solution is: $x_h(t)=c_1 e^{-\alpha t}$

Next, we find any solution $x(t)$ that satisfies the differential equation. For a constant coefficient, we use $x_p(t)=\beta/\alpha$ (check to verify that it satisfies the original differential equation).

We then calculate

$$x(t)=x_h(t)+x_p(t)$$

$$x(t)=c_1 e^{-\alpha t}+\frac{\beta}{\alpha}$$

Knowing one boundary condition (the value of *x(t)* for some *t*), plug in the value of *x(t)* and *t*, and then you can find c_1. With that in mind, let's continue to the solutions...

Problem 1. The network shown in Figure 1 has zero initial state. That is, the inductor current $i(t{=}0^-) = 0$. At $t = 0$, the voltage source produces an impulse of area 100, as shown in Figure 2.

a.) Derive the differential equation for $i(t)$ as a function of $V(t)$.

b.) Find the inductor current $i(t)$ at $t = 0^+$.

c.) Find the inductor current $i(t)$ at t = 1000 through physical, rather than mathematical, reasoning.

d.) Write an explicit equation for $i(t)$.

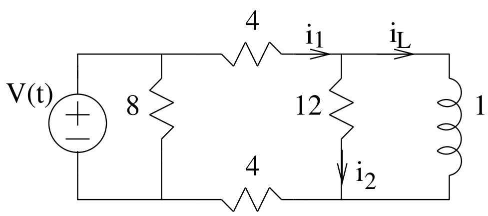

Figure 1. Circuit for Problem 1

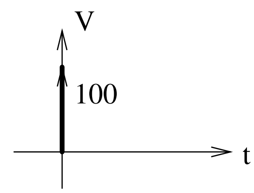

Figure 2. Input

Solution:

It is probably most intuitive to solve for the circuit's step response, and then take the time derivative of the step response to find the impulse response.

To start, we apply KCL at the node at which current enters the inductor:

(1) $i_1 = i_2 + i_L$

Notice that the current i_1 flows through both 4 Ω resistors; this leaves only the voltage across the 12 Ω resistor and the inductor (in parallel), as well as i_1 as unknowns. Using KVL:

(2) $v_L = V_I - 2(i_1 * 4\,\Omega)$

with the constitutive relations:

$$v_L = L\frac{di_L}{dt} \qquad\qquad v_L = i_r * 12\,\Omega$$

solving (2) for i_1:

$$i_1 = \frac{L\frac{di_L}{dt} - V}{-8\,\Omega} = \frac{V - L\frac{di_L}{dt}}{8\,\Omega}$$

then plug everything into (1):

$$\frac{V - L\frac{di_L}{dt}}{8\,\Omega} = i_L + \frac{1}{12\,\Omega}\left(V - 2\left(4\,\Omega\frac{V - L\frac{di_L}{dt}}{8\,\Omega}\right)\right)$$

and then some algebra gives (answering part a––we have an equation relating i(t) and V(t)):

$$\frac{5}{3}L\frac{di_L}{dt} + 8\,\Omega * i_L = V$$

$$\frac{di_L}{dt} + \frac{24\,\Omega}{5\,L} i_L = \frac{60\,V}{L}$$

Since $L = 1$ H, we have

$$\frac{di_L}{dt} + \frac{24}{5\,s} i_L = 60\,A/s$$

Solving this using our now old hat formula, we have:

$$i_L(t) = c_1 e^{-(24/5\,s)\,t} + 12.5\,A$$

The continuity of current in the step response gives $i_L(t=0^+) = 0$. Therefore, $c_1 = -12.5$ A, and the total step response is:

$$i_{L,\,step\,response}(t) = 12.5\,A + (-12.5\,A)e^{-(24/5\,s)\,t}$$

However, now we differentiate with respect to time to find the response to the given impulse. We find that

$$t \leqslant 0 \qquad i_L(t) = 0$$

$$0 \leqslant t \qquad i_L(t) = 60\,e^{-(24/5\,s)\,t}$$

This is the answer to part d.) We now evaluate i_L at $t = 0^+$ and find that $i_L(t=0^+) = 60$ A. This is the answer to part b.) Finally, to answer part c.), one must think what happens to the inductor as it experiences the impulse from the voltage source. Instantaneously, the current in the inductor is forced to some value, as the infinite voltage creates an instantaneous jump in the flux linkage in the inductor. This stores energy in the inductor's magnetic field, and once the source is turned off, the energy is slowly dissipated in the circuit's resistors. We therefore expect that $i_L(t=1000) \approx 0$. This is the answer to part c.)

Problem 2. A + L, page 693, #19. We are given the circuit in Figure 3 below, the initial condition $i(t=0)=2mA$, and the inputs $e_1(t)$ and $e_2(t)$.

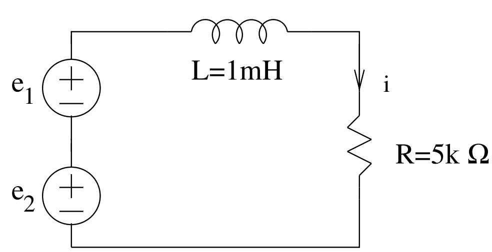

Figure 3. Circuit for Problem 2

a.) Let $i_2(t)$ and $i_1(t)$ be responses from $e_1(t)$ and $e_2(t)$, applied one at a time, using the given initial condition. Plot $i_2(t)$ and $i_1(t)$. Is it true that $i_2(t) = 2i_1(t)$ for all time $t \geq 0$?

b.) Consider now the zero–state responses due to $e_1(t)$ and $e_2(t)$; call them $i_1'(t)$ and $i_2'(t)$, respectively. Plot $i_1'(t)$ and $i_2'(t)$. Is it true that $i_1'(t) = 2\, i_2'(t)$ for all time $t \geq 0$?

Solution to part a.): Considering the response to e_1 alone, the final inductor current (as $t \Rightarrow \infty$) is simply $i_L(t \rightarrow \infty) = 10V/5k\Omega = 2\,mA$, since the inductor acts like a short circuit a long time after the transient. Also, we know that the equation $i_L(t)$ for $t \geq 0$ follows the formula

(1) $$i_L(t) = I_{Final} + (I_{Initial} - I_{Final})e^{-t/\tau}$$

where the coefficient in front of the exponential captures the exponential approach of the inductor current to its final value as well as the exponential decay of the initial condition. Here, however, $I_{Initial} = I_{Final} = 2\,mA$, so for $t \geq 0$, the coefficient of the exponential term is zero:

$$i_1(t) = 2\,mA$$

We can use the same method to solve for $i_2(t)$. For very long times, the inductor acts like a short circuit, and so we have $i_L(t \rightarrow \infty) = 20V/5k\Omega = 4\,mA = I_{Final}$. Using the same method (1) as above, we have

$$i_2(t) = 4\,mA + (2\,mA - 4\,mA)e^{-t/\tau}$$

with $\tau = L/R = 0.2\,\mu\,sec$. The plots of these two currents are shown in Figure 4; clearly, it is not the case that for all $t \geq 0$, $i_2(t) = 2\, i_1(t)$. As the initial transient dies out, this relation is approached, but it is not true for all $t \geq 0$.

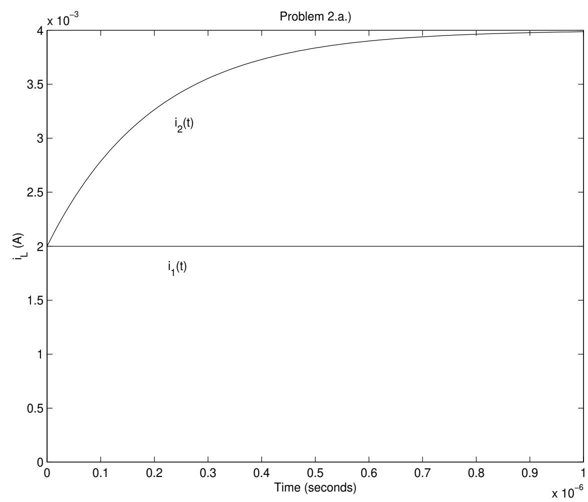

Figure 4. $i_1(t)$ and $i_2(t)$

Solution to part b.): We now solve for the zero state responses, that is, $i_1(t)$ and $i_2(t)$, assuming that there is no initial state ($i(0) = 0 = I'_{Initial}$). Our final inductor currents for $i_1(t)$ and $i_2(t)$ as $t \to \infty$, we calculate the same values as from part a.):

$$i'_1(t \to \infty) = 10\,V/5\,k\Omega = 2\,mA = I'_{Final_1}$$
$$i'_2(t \to \infty) = 20\,V/5\,k\Omega = 4\,mA = I'_{Final_2}$$

Using (1) from part a.), we have

$$i'_1(t) = I'_{Final_1} + (I'_{Initial} - I'_{Final_1})e^{-t/\tau}$$
$$i'_2(t) = I'_{Final_2} + (I'_{Initial} - I'_{Final_2})e^{-t/\tau}$$

This results in the two equations for $t \geq 0$:

$$i'_1(t) = 2\,mA(1 - e^{-t/\tau})$$
$$i'_2(t) = 4\,mA(1 - e^{-t/\tau})$$

These two currents are plotted in Figure 5; we see that algebraically it is true for $t \geq 0$ that $i_1'(t) = 2\,i_2'(t)$.

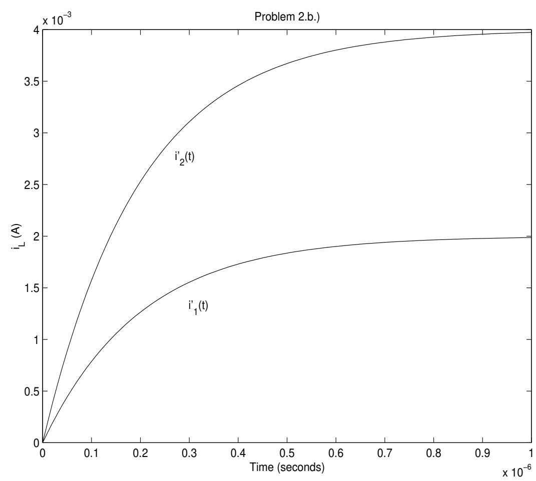

Figure 5. i_1'(t) and i_2'(t)

Problem 3. In the circuit given in Figure 6, the threshold voltage V_t = 0 V, and K = 2 mA/V^2. The input V_I is depicted in Figure 7.

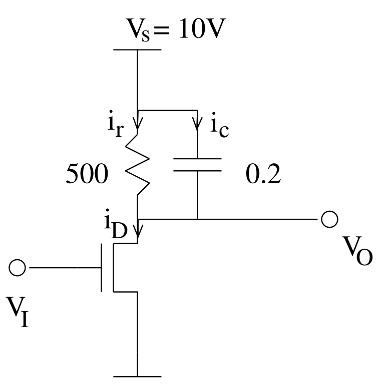

Figure 6. Circuit for problem 3

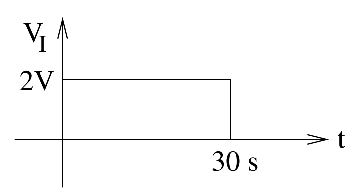

Figure 7. Input for problem 3

a.) Does the MOSFET operate in the saturated regime? When?

b.) Write an explicit expression for the output voltage for $t \leq 30$ μsec.

c.) Write a differential equation that governs the value of the output for time $t \geq 30$ μsec.

d.) Write an explicit expression for the output voltage for time $t \geq 30$ μsec.

Solution:

Our initial conditions are that $i_D = 0$, and thus $i_r = i_c = 0$. (The MOSFET can't be in linear mode, since $V_{ds} > V_{gs} - V_t$. $V_{gs} = 0$ as well, so the drain current must be 0.) The voltage across the capacitor must be zero as well, because if it weren't, current would be flowing through the resistor (since they are in parallel, a voltage difference across the capacitor is a voltage difference across the resistor), and this current would have to be charging the capacitor back to zero voltage across it. More on that later...

At $t = 0$:

$$(V_{ds}=10V)>(V_{gs}-V_t=2V) \Rightarrow i_D=\frac{K}{2}(V_{gs}-V_t)^2=4\,mA$$

Now we apply KCL to the drain node:

$$i_D=i_r+i_c$$

We have the constitutive relations:

$$i_r=\frac{10V-V_o}{500\,\Omega} \qquad i_c=C\frac{d}{dt}(10V-V_o)=-C\frac{d}{dt}V_o$$

Giving, after substitution:

$$\frac{10V-V_o}{500\,\Omega}+0.2\,\mu F\left(\frac{-dV_o}{dt}\right)=4\,mA$$

Simple algebra gives:

$$10^{-6}\frac{dV_o}{dt}+\frac{V_o}{100\,sec}-\frac{1\,Volt}{10\,sec}=-20\frac{mV}{sec}$$

$$\frac{dV_o}{dt}+1*10^4\,sec^{-1}V_o-1*10^5\,V/sec=-2*10^4\,V/sec$$

$$\frac{dV_o}{dt}+1*10^4\,sec^{-1}V_o=8*10^4\,V/sec$$

Now we can use the differential equation from the calibration quiz (you remember that, don't you? =) again. We find, using this formula, that:

$$V_o(t) = c_1 e^{-t/100\mu sec} + \frac{8*10^4 V/sec}{\frac{1}{100\mu sec}} = c_1\, e^{-t/100\mu sec} + 8V$$

Using the boundary condition at t = 0, $V_o(t=0^+)$ = 10 V, we find that c_1 = 2. Thus our total solution is:

$$0 \leqslant t \leqslant 30\mu sec \;\; : \qquad V_o(t) = 8V + (2V)e^{-t/100\mu sec}$$

In this time interval, the MOSFET operates in the saturated regime.

Now, at t = 30 µsec, the input drops to 0 Volts once more. The MOSFET goes back into cutoff mode and stops carrying current (i_D = 0). The circuit can effectively be redrawn as shown in Figure 8.

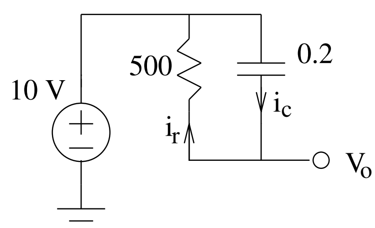

Figure 8. Equivalent circuit for t >= 30

Unlike for t < 0, when there is no voltage drop across the resistor and capacitor, the initial condition for this new circuit is that the voltage drop is $V_o(t=30\ \mu sec)$. Call this value α; numerically, α=9.48 V.

Because a voltage drop across the resistor exists, there is a current flowing through it, of value:

$$i_r = \frac{V_o - 10V}{500\Omega}$$

KCL at the V_o node gives $i_r = i_c$, and so we find a new differential equation:

$$\frac{V_o - 10V}{500\Omega} = -C\frac{dV_o}{dt}$$

Rearranging, we find that:

$$\frac{dV_o}{dt} + \frac{V_o}{100\mu sec} = \frac{10V}{100\mu sec}$$

We use the now very familiar solution to a 1st order differential equation with constant coefficients, and find:

$$(t' = t - 30\mu sec) \qquad\qquad V_o(t') = c_1 e^{-t'/100\mu sec} + 10V$$

From the boundary condition $V_o(t' = 0) = \alpha$, we calculate that $c_1 = \alpha - 10V$. Our whole solution for $t \geq 30$ μsec is then:

$$30\,\mu\, sec \leqslant t \qquad\qquad V_o(t) = 10V - 0.52V\, e^{-(t-30\,\mu\, sec)/100\,\mu\, sec}$$

Problem 4. We are given the circuit in Figure 9 and the input $v_s(t) = 2 + 0.01u(t)$. We measure $v_c(t)$ as the voltage from the drain terminal to the gate terminal of the MOSFET.

$R_L = 10\ k\Omega$
$C = 1\ \mu F$
$R_S = 1\ k\Omega$
$V_T = 1\ V$
$k = 2\ mA / V$
$V_{DD} = 15\ V$

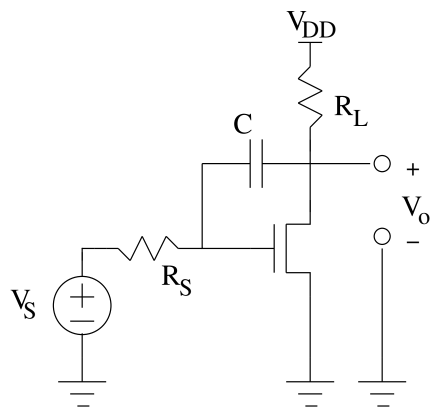

Figure 9. Circuit for Problem 4

a.) Calculate the bias value of v_c for t < 0. You may assume that the capacitor has the voltage that would appear between the drain and the gate if the capacitor were not present.

Solution: If the capacitor were absent from the circuit, there would be no current flowing through R_S, so the voltage at the gate would be 2 V. Thus, the gate to source voltage is greater than $V_T = 1$ V, so the transistor is in saturation or in linear mode. Assuming that the MOSFET is in saturation, we have

$$I_D = \frac{K}{2}(V_{GS} - V_T)^2 = 1\, mA$$

from which that

$$V_{OUT} = V_{DD} - I_D R_L = 15\, V - 1mA * 10k\,\Omega = 5\, V$$

This gives $(V_{DS} = 5\, V) > (V_{GS} - V_T = 1\, V)$, verifying that the MOSFET is saturated.

b.) Make an incremental model of the circuit for the bias conditions you found in part a.)

Solution:

With a bias gate–source voltage V_{GS}=2 V, we have

$$g_m = K(V_{GS} - V_T) = 2\, mA/V$$

The incremental circuit is shown in Figure 10:

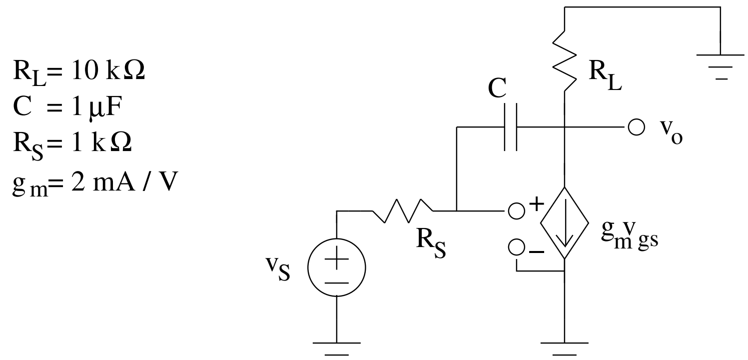

Figure 10. Incremental Model for Problem 4

c.) Determine the differential equation for v_o as a function of v_s.

Solution:

This might seem quite complicated at first glance, because we're not solving for v_c, first of all; secondly, the voltages at the nodes to which the capacitor is connected are not simply related; thirdly, there is a dependent source in the circuit.

As always, we can revert to KCL and KVL though; KCL at the v_{out} node says that the current into the node through the capacitor equals the current flowing out through the other two paths: the dependent source, and the load resistor:

(1) $$C\frac{d(v_{gs}-v_{out})}{dt}=g_m v_{gs}+\frac{v_{out}}{R_L}$$

Since we don't want any v_{gs} terms to appear in our equation, we would like to find an expression for v_{gs} in terms of the quantities we do want to appear, like v_s and v_{out}. We can do this by relating the current through the capacitor to the current through the source resistor; by KCL, the currents are equal:

(2) $$\frac{v_s-v_{gs}}{R_s}=C\frac{d(v_{gs}-v_{out})}{dt}$$

From this we can equate (1) and (2) and find:

$$\frac{v_s}{R_s}-\frac{v_{out}}{R_L}=(g_m+\frac{1}{R_s})v_{gs}$$

Thus:

(3) $$v_{gs}=\frac{v_s/R_s-v_{out}/R_L}{g_m+1/R_s}$$

Now that we have an expression for v_{gs}, we can reintroduce it into either (1) or (2) to find the desired differential equation. Substitution of (3) into (2), with the introduction of

$$\alpha=g_m+1/R_s$$

gives us:

$$\frac{v_s}{R_s} - \frac{1}{R_s}\left(\frac{v_s}{\alpha R_s} - \frac{v_{out}}{\alpha R_L}\right) = \left(\frac{C}{\alpha R_s}\right)\frac{d v_s}{dt} - \left(\frac{C}{\alpha R_L}\right)\frac{d v_{out}}{dt} - C\frac{d v_{out}}{dt}$$

Rearranging, we have:

(4) $$\frac{1-\alpha R_s}{\alpha R_s^2} v_s + \left(\frac{C}{\alpha R_s}\right)\frac{d v_s}{dt} = \left(\frac{1}{\alpha R_L R_s}\right) v_o + C\left(1+\frac{1}{\alpha R_L}\right)\frac{d v_o}{dt}$$

d.) Give an algebraic expression for the incremental output voltage as a function of time for t>0.

Solution:

Since $v_s(t) = 0.01\, u(t)$, we let $\gamma = 0.01$, and the differential equation (4) becomes

$$\gamma\frac{1-\alpha R_s}{\alpha R_s^2} u(t) + \gamma\left(\frac{C}{\alpha R_s}\right)\delta(t) = \left(\frac{1}{\alpha R_L R_s}\right) v_o + C\left(1+\frac{1}{\alpha R_L}\right)\frac{d v_o}{dt}$$

Solving this equation can be done by finding the response to the step input and the impulse input. We can then add the responses since there is no initial condition: v_o=0. The response to the step input can be solved with

$$\gamma\frac{1-\alpha R_s}{\alpha R_s^2} = \left(\frac{1}{\alpha R_L R_s}\right) v_{o_{STEP}} + C\left(1+\frac{1}{\alpha R_L}\right)\frac{d v_{o_{STEP}}}{dt}$$

Using an incredibly familiar relation, we know that we have

$$v_{o_{STEP}}(t) = \frac{\gamma(1-\alpha R_s) R_L}{R_s}\left(1 - e^{-\frac{t}{C(\alpha R_L R_s + R_s)}}\right)$$

We then find the response to the impulse by solving

$$\frac{\gamma C}{\alpha R_s}\delta(t) = \left(\frac{1}{\alpha R_L R_s}\right) v_{o_{IMPULSE}} + C\left(1+\frac{1}{\alpha R_L}\right)\frac{d v_{o_{IMPULSE}}}{dt}$$

Integrating from t=0$^-$ to t=0$^+$, we have the response to the impulse:

$$v_{o_{IMPULSE}}(t) = \left(\frac{\gamma}{\alpha R_s\left(1+\frac{1}{\alpha R_L}\right)}\right) e^{\frac{-t}{C(\alpha R_L R_s + R_s)}}$$

And summing terms, we find:

$$v_o(t) = \frac{\gamma(1-\alpha R_s) R_L}{R_s}\left(1 - e^{\frac{-t}{C(\alpha R_L R_s + R_s)}}\right) + \left(\frac{\gamma}{\alpha R_s\left(1+\frac{1}{\alpha R_L}\right)}\right) e^{\frac{-t}{C(\alpha R_L R_s + R_s)}}$$

e.) See Figure 11 below.

Numerically, for t > 0,

$$v_o(t) = (-0.2\,V) e^{-t/.031\,sec} + (0.0032\,V) e^{-t/.031\,sec}$$

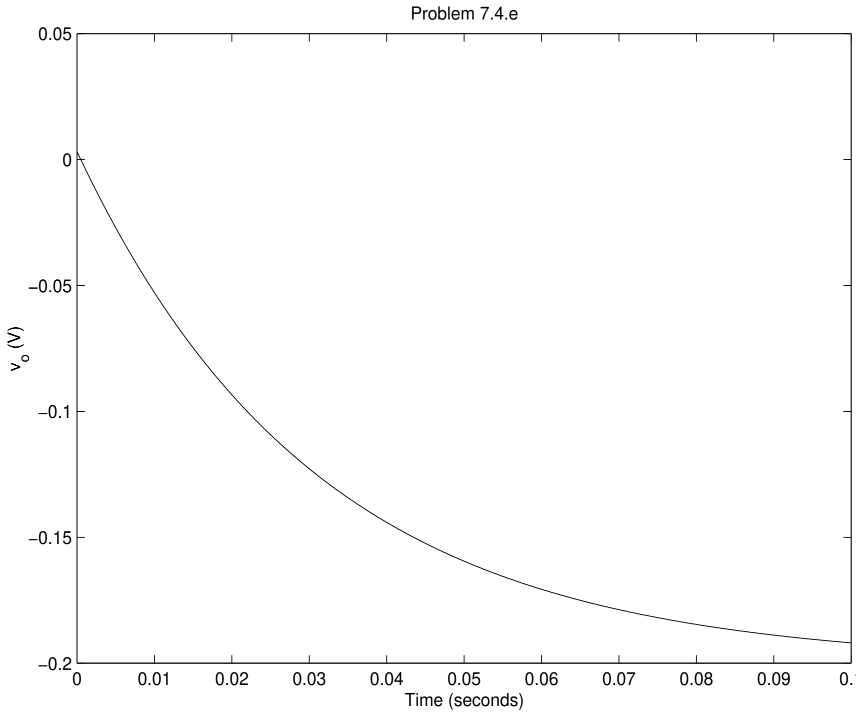

Figure 11. $v_o(t)$