6.002 Fall 2001 Problem Set 8 Solutions

Problem 1.

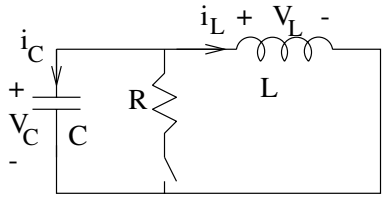


Figure 1. Circuit for Problem 1

Given the circuit in Figure 1 (S_1 has closed) and zero initial condition, calculate:

a.) the time response of the coil current after the switch S_1 is closed and before the capacitor voltage reaches zero.

b.) the peak coil current for V_x = 1000 V.

c.) the time t_1 at which the capacitor voltage reaches zero and S_2 is closed.

d.) the time response of the coil current after the switch S_2 is closed.

Solution to part a.): We know $i_L(t<0)=0$, as our initial condition, that $v_c(t)=v_L(t)$ and that $i_c(t)=-i_L(t)$ while S_1 is closed and S_2 is open. Thus

$$C\frac{dv_L}{dt}+i_L=0$$

Substituting the constitutive relation for the inductor we have

$$LC\frac{d^2i_L}{dt^2}+i_L=0$$

Assuming a solution of the form $i_L(t)=A_1e^{s_1t}+A_2e^{s_2t}$ (there is no particular solution in this instance because the original differential equation is homogeneous), and plugging this solution into the differential equation we find that

$$LCs_1^2+1=0$$

$$LCs_2^2+1=0$$

Thus, $s_{1,2}=\pm j\sqrt{\frac{1}{LC}}$ and if we let $\omega_0=\sqrt{\frac{1}{LC}}$,

$$s_1=j\omega_0$$

$$s_2=-j\omega_0$$

Now we find A_1 and A_2 with our initial conditions.

$$v_c(t=0) = V_x \quad \rightarrow \quad L\frac{di_L}{dt} = L\,j\omega_0(A_1 - A_2) = V_x$$

$$i_L(t=0) = 0 \quad \rightarrow \quad A_1 + A_2 = 0 \quad \rightarrow \quad A_2 = -A_1$$

Solving these two equations gives

$$A_1(2\,L\,j\omega_0) = V_x \quad \rightarrow \quad A_1 = \frac{1}{\sqrt{\frac{L}{C}}}\frac{V_x}{2\,j}$$

Using the relation between A_1 and A_2, we have

$$i_L(t) = V_x\left(\sqrt{\frac{C}{L}}\right)\left(\frac{e^{j\omega_0 t} - e^{j\omega_0 t}}{2\,j}\right) = \left(V_x\sqrt{\frac{C}{L}}\right)\sin(\omega_0 t)$$

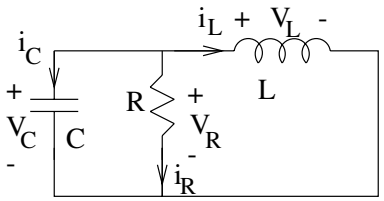
for $0 \le t < t_1$, where t_1 is the smallest $t_1 > 0$ such that $v_c(t_1) = 0$.

Solution to part b.): We know that $v_c(t) = v_L(t) = L\frac{di_L}{dt} = LV_x\sqrt{\frac{C}{L}}\omega_0\cos(\omega_0 t)$, and so the smallest $t > 0$ such that $v_c(t) = 0$ is at $\omega_0 t_1 = \pi/2$. At $t = t_1$, $i_{L,peak} = V_x\sqrt{\frac{C}{L}}$. If V_x = 1000 V, $i_{L,\,peak}$ = 4.045 * 10^3 A.

Another approach is to use conservation of energy, which is easy because there is no dissipative element in the system until S_2 closes. Since $E_{stored} = \frac{1}{2}CV_x^2$ at t=0, and v_c=0 and i_L=$i_{L,peak}$ when the switch S_2 closes, we have $\frac{1}{2}CV_x^2 = \frac{1}{2}Li_{L,peak}^2$.

Solution to part c.): From the relation $\omega_0 t_1 = \pi/2$, we have $t_1 = \pi/2\,\omega_0 \approx 70\,\mu s$

Solution to part d.): To find $i_L(t)$ once the switch S_2 is closed, we write new KVL and KCL equations according to the circuit in Figure 2:



C = 180 uF
R = 80 mOhm
L = 11 uH
$V_c(t = t_1) = 0$
$i_L(t= t_1) = i_{L,peak}$

Figure 2. Equivalent circuit after S_2 closes

KCL: (1) $$i_c(t) + i_R(t) + i_L(t) = 0$$

KVL: (2) $$v_c(t) = v_R(t) = v_L(t)$$

Recasting our variables in terms of i_L, we find that

$$i_c = C\frac{dv_c}{dt} = C\frac{dv_L}{dt} = LC\frac{d^2 i_L}{dt^2}$$

and

$$i_R = \frac{v_R}{R} = \frac{L}{R}\frac{di_L}{dt}$$

Combining these with (1) gives us the differential equation:

$$i_L + \frac{L}{R}\frac{di_L}{dt} + LC\frac{d^2 i_L}{dt^2} = 0$$

In standard form,

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC}i_L = 0$$

To make math easier, we'll use a change of variable: $t' = t - t_1$

Then for $t > t_1$,

$$i_L(t') = A_1 e^{s_1 t'} + A_2 e^{s_2 t'}$$

Plugging this into the differential equation gives:

$$s_{1,2} = \frac{\frac{-1}{RC} \pm \sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}}{2}$$

Calculating the value of the radicand, we find that it is positive, so the circuit does not exhibit oscillatory behavior; it is overdamped. We calculate that

$$\tau_1 = (-s_1)^{-1} = 121\,\mu s$$
$$\tau_2 = (-s_2)^{-1} = 16.3\,\mu s$$

Now we need to calculate A_1 and A_2 using the initial conditions.

$$i_L(t'=0) = A_1 + A_2 = V_x\sqrt{\frac{C}{L}}$$

$$v_c(t'=0) = -\frac{A_1}{\tau_1} + -\frac{A_2}{\tau_2} = 0$$

This gives finally

$$A_1 = 4.67*10^3\,A$$

$$A_2 = -630.7\,A$$

So the solution for t > t_1 is

$$i_L(t) = (4.67*10^3\,A)e^{-(t-t_1)/121\,\mu s} - (630.7\,A)e^{-(t-t_1)/16.3\,\mu s}$$

Problem 2.

Given the two circuits (A) and (B) shown in Figure 3, find $v_c(t)$ for t > 0 for both. Identify the effects of the inductance on the natural frequencies, approximate behavior for small *t*, and asymptotic behavior. There is zero initial state in both circuits ($v_c(t=0^-)=0$, and $i_L(t=0^-)=0$).

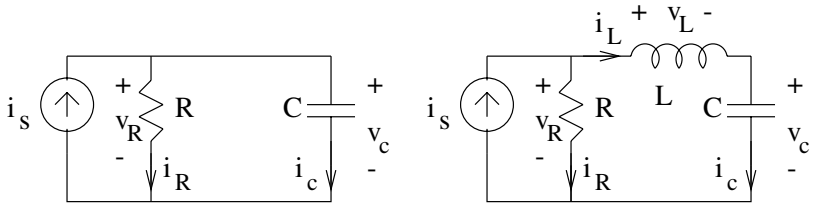


Figure 3. Circuits for Problem 2

Solution for circuit (A):

From KCL at the top node,

$$i_s(t) = i_R(t) + i_c(t)$$

$$i_s(t) = \frac{v_c}{R} + C\frac{dv_c}{dt}$$

giving the differential equation

$$\frac{dv_c}{dt} + \frac{1}{RC}v_c = \frac{1}{C}I$$

This gives us the well–known answer for t > 0:

$$v_c(t) = I\,R\,(1 - e^{-t/RC})$$

The natural frequency of this circuit is just the decay of time constant RC.

Around t = 0, the Taylor expansion $e^{-x} \approx 1-x$ gives

$$v_c(t) \approx IR - IR(1 - t/RC) = \frac{I}{C}t$$

so the capacitor voltage is, to first order, linear in time.

Solution for circuit (B):

Applying KCL at the top node and KVL around the R–LC loop give us

(1) $$i_s(t) = i_L(t) + i_R(t)$$

(2) $$v_R(t) = v_L(t) + v_c(t)$$

Since the capacitor and inductor are in series, $i_L = i_c$; combining that with the constitutive relations $v_L = L\frac{di_L}{dt}$ and $i_c = C\frac{dv_c}{dt}$ gives

(3) $$v_L = LC\frac{d^2 v_c}{dt^2}$$

Substituting (3) into (2) gives

(4) $$v_R = LC\frac{d^2 v_c}{dt^2} + v_c$$

and substituting constitutive relations into (1) gives

$$i_s = C\frac{dv_c}{dt} + \frac{v_R}{R}$$

or

$$i_s = C\frac{dv_c}{dt} + \frac{LC}{R}\frac{d^2 v_c}{dt^2} + \frac{v_c}{R}$$

Rearranging this into standard form,

(5) $$\frac{d^2 v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{1}{LC}v_c = \frac{R}{LC}i_s(t)$$

To solve the homogeneous equation (setting the right hand side to 0), we assume a solution that is a sum of two exponential terms of the form $A_1 e^{s_1 t} + A_2 e^{s_2 t}$.

Substituting this form into the homogeneous equation, we find that

This gives us two roots by the $s_{1,2}^2 + \frac{R}{L}s_{1,2} + \frac{1}{LC} = 0$ quadratic equation,

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$

combining, we have

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2C - 4L}{L^2C}}$$

The radicand is *real* because we are given that $L \ll RC^2/4$. Thus, we can simplify the expression for the natural frequencies to

$$s_{1,2} \approx -\frac{R}{2L} \pm \frac{R}{2L}(1 - \frac{2L}{R^2C})$$

Then $s_1 = \frac{-1}{RC}$ and $s_2 = \frac{-R}{L} + \frac{1}{RC}$. So we have

(6) $$v_c(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + v_p(t)$$

and here we may choose $v_p(t) = IR$ since that satisfies the differential equation (5). The task is then to calculate A_1 and A_2 from the initial conditions. Since $v_c(t<0)=0$, we obtain one equation:

$$A_1 e^0 + A_2 e^0 + IR = 0$$

or (7) $$A_1 + A_2 = -IR$$

and since $i_L(t<0)=0$, we obtain the other needed equation:

$$i_L = i_c = C\frac{dv_c}{dt} = 0$$

using (6), this results in:

(8) $$A_1 s_1 + A_2 s_2 = 0 \quad \rightarrow \quad A_2 = -A_1 \frac{s_1}{s_2}$$

Combining (7) and (8), and multiplying numerator and denominator by RLC we find that

$$A_1 = \frac{-IRs_2}{s_2 - s_1} = \frac{-IR(-R^2C + L)}{-R^2C + 2L}$$

$$A_2 = \frac{-IRL(\cancel{-R^2C + L})}{(\cancel{-R^2C + L})(-R^2C + 2L)}$$

Then

$$v_c(t) = \frac{-IR(-R^2C + L)}{-R^2C + 2L} e^{-t/RC} + \frac{-I\,R\,L}{-R^2C + 2L} e^{(-R/L+1/RC)t} + IR$$

This is the complete solution to $v_c(t)$ for t > 0. The effect of adding the inductor is seen in the second term: a quickly decaying exponential with coefficient proportional to L (ie, of small magnitude for small L).

To find the behavior around t = 0, we again use a Taylor expansion, and find that for t≈0,

$$v_c(t) \approx A_1(1 + s_1 t) + A_2(1 + s_2 t) + IR$$

But since $A_1 + A_2 + IR = 0$ and $A_1 s_1 + A_2 s_2 = 0$,

$$v_c(t) \approx 0$$

To first order, the capacitor voltage is constant at zero immediately after the step is applied, since the inductor initially acts like an open circuit.

Asymptotically, the circuits behave identically: in both cases, the capacitor voltage approaches *IR* as t goes to infinity.

Problem 3.

We have two charging circuits, shown in Figure 4; $v_c(t)=0$ for both, and at $t=0$, switch *S* closes for both circuits, and is re−opened when $i_c(t)$ reaches 0 after $t=0^+$.

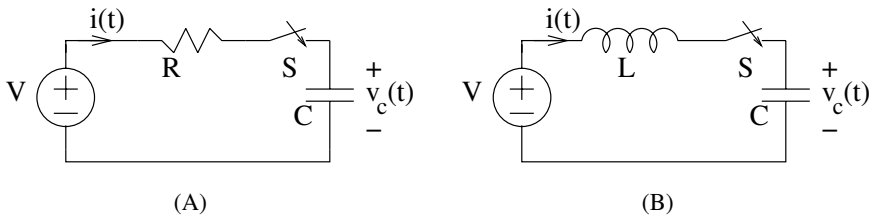


Figure 4. Circuits for Problem 3

a.) sketch $i_c(t)$ and $v_c(t)$ for circuit A. How much energy is stored on the capacitor? How much is dissipated in charging the capacitor?

Solution: Both $i_c(t)$ and $v_c(t)$ should be familiar plots by now (see Figure 5).

The expression for $v_c(t)$ for t > 0 and the expression for $i_c(t)$ for t > 0 are of course:

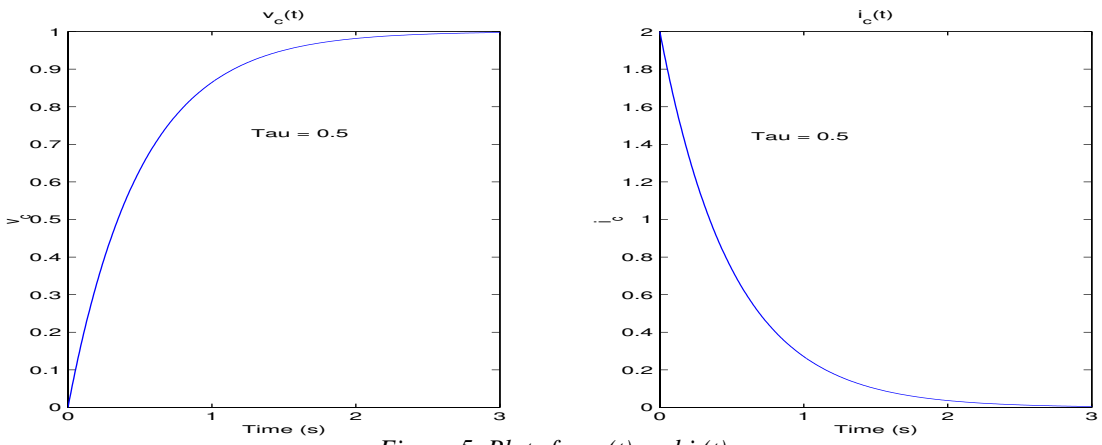


Figure 5. Plots for $v_c(t)$ and $i_c(t)$

The energy stored on the capacitor is the familiar

$$E = \int_0^\infty v_c(t)\, i_c(t)\, dt = \frac{1}{2} C V^2$$

and the energy dissipated in charging the capacitor is still

$$E = \int_0^\infty v_r(t)\, i_r(t)\, dt = \int_0^\infty (i_c(t) * R)\, i_c(t)\, dt$$

$$E = R \int_0^\infty \left(\frac{V}{R}\right)^2 e^{-2\frac{t}{RC}}\, dt = \frac{V^2}{R}\left[e^{\frac{-2t}{RC}}\right]_0^\infty = \frac{1}{2} C V^2$$

b.) Since the switch is open for $t < 0$, we know that $i_c(t) = i_L(t) = 0$ for $t < 0$. This is our second initial condition for the LC circuit. Taking a KVL loop, we have

$$V = v_L + v_c = L\frac{d i_L}{dt} + v_c = CL\frac{d^2 v_c}{dt^2} + v_c$$

and this is the governing differential equation for the circuit. We assume our solution is of the form $v_c t = v_{c,h}(t) + v_{c,p}(t)$ with $v_{c,p}(t) = V$ and $v_{c,h}(t) = A_{1e}^{s_1 t} + A_{2e}^{s_2 t}$. Substituting these equations into the differential equation, we find that

$$LC\, s_1^2 + 1 = 0$$

$$LC\, s_2^2 + 1 = 0$$

In other words, with $\omega_0 = \frac{1}{\sqrt{LC}}$, we have $s_1 = j\omega_0$ and $s_2 = -j\omega_0$.

Now we want to find A_1 and A_2 using the initial conditions. We have

$$v_c(t=0) = \left[(A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t} + V)\right]_{t=0} = 0$$

$$i_L(t=0) = C\left[(A_1 j\omega_0) e^{j\omega_0 t} + (-A_2 j\omega_0) e^{-j\omega_0 t}\right]_{t=0} = 0$$

This gives $A_1 = -V/2$, $A_2 = -V/2$. Substituting these expressions into the original equations for $v_c(t)$ gives

$$v_c(t) = V\,(1 - \cos(\omega_0 t))$$

$$i_c(t) = CV\omega_0 \sin(\omega_0 t)$$

When the switch opens, $i_c(t_0) = 0$ for some $t_0 > 0^+$. The first zero crossing of $i_c(t)$ is at $t_0 = \pi/\omega_0$, at which time $v_c(t_0) = 2V$.

The energy stored in the capacitor after the switch opens is .

$$E_{stored} = \frac{1}{2} C (2V)^2 = 2CV^2$$

The energy dissipated can be calculated via energy conservation:

$$E_{dissipated} = E_{supplied} - E_{stored} = \int_0^{\pi/\omega_0} V\, i_c(t)\, dt - 2CV^2$$

$$E_{dissipated} = CV^2\omega_0 \int_0^{\pi/\omega_0} \sin(\omega_0 t)\, dt - 2CV^2$$

$$E_{dissipated} = 2CV^2 - 2CV^2 = 0$$

A plot of the capacitor voltage and inductor current for circuit (B) resembles Figure 6:

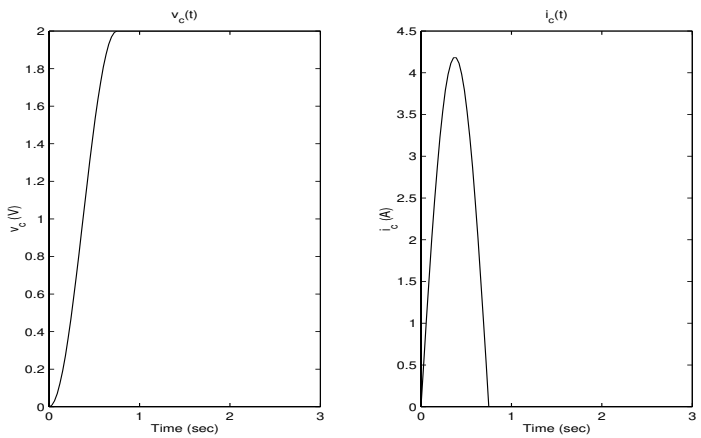


Figure 6. $v_c(t)$ and $i_c(t)$ in inductor charging circuit

Problem 4.

Calculate the impedance of the United Chemi–Con U767D Series 35 V electrolytic capacitor, for both the ideal case and for the non–ideal model shown in Figure 7 below.

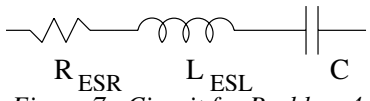


Figure 7. Circuit for Problem 4

Here, C = 2200 μF, R_{ESR} = 20 mΩ, and L_{ESL} = 18nH, at the following frequencies: 60 Hz, 60 kHz, and 6 Mhz. How do the ideal and real impedances compare?

Solution:

The impedances as functions of frequency are:

$$Z_{REAL}(j\omega) = \frac{1}{j\omega C} + j\omega L_{ESL} + R_{ESR}$$

$$Z_{IDEAL}(j\omega) = \frac{1}{j\omega C}$$

At 60 Hz (ω = 120 π):

$$Z_{REAL} = .02 - 1.21j = 1.21e^{-1.55j}$$
$$Z = 0 - 1.21j = 1.21e^{-j\pi/2}$$

At 60 kHz (ω = 120 π ∗ 10^3):

$$Z_{REAL} = .02 + .0056j = .021e^{0.272j}$$
$$Z = 0 - 0.0012j = .0012e^{-j\pi/2}$$

At 6 MHz (ω = 1.2 π ∗ 10^6):

$$Z_{REAL} = 0.02 + 0.68j = .68e^{1.54j}$$
$$Z_{IDEAL} = 0 - 1.21*10^{-5}j = 1.21*10^{-5}e^{-j\pi/2}$$

The impedance of the ideal capacitor has a constant phase, and the magnitude of the impedance approaches 0 as ω approaches ∞. In contrast, the impedance of the real capacitor has a phase close to –π/2 at low frequencies, with the magnitude of the impedance decreasing; in other words, at low frequencies the actual element behaves like a capacitor. At medium frequencies, the phase and magnitude are constant, and the element behaves like a resistor. At high frequencies, the phase shift is close to +π/2, and the magnitude of the impedance increases with frequency––the device behaves like an

inductor. Finally, the impedance has a minimum magnitude at

$$\omega = \omega_0 = \frac{1}{\sqrt{L_{ESL} C}}$$

Figure 8 shows the magnitude and phase of the impedance as a function of ω.

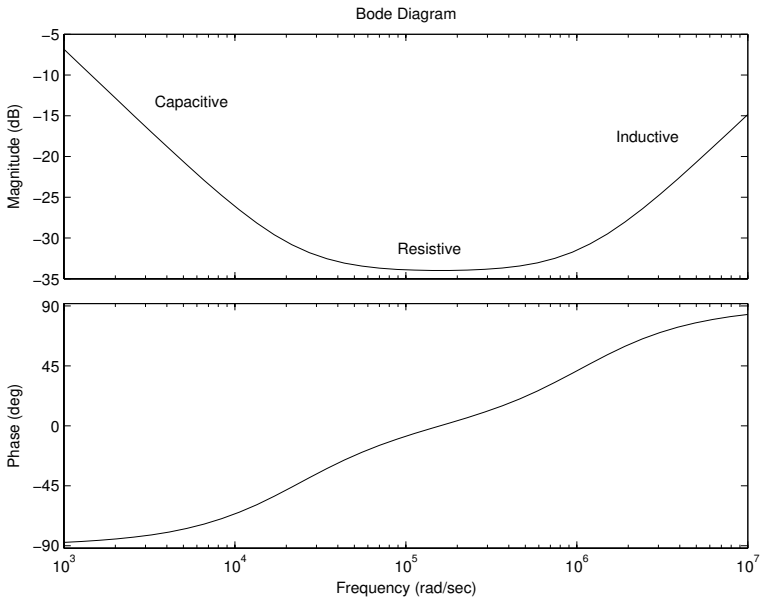


Figure 8. Impedance as a function of frequency

Problem 5.

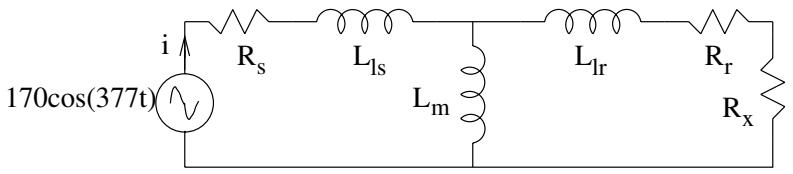


Figure 9. Circuit for Problem 5

Given the circuit in Figure 9, with the component values R_s = 0.08Ω, L_{ls} =1mH, L_m = 40mH, L_{lr} = 1mH, R_r = 0.1Ω, R_x = 33Ω, and the input voltage *v(t)* = 170 cos (377 t), find the steady state current into the motor. Also, find the time average power delivered by the source.

Solution:

Since $v(t) = \Re\{V_o\, e^{j\omega t}\}$, we know $\omega_0 = 377 \approx 120\,\pi$. Using the constitutive laws for resistors and inductors in steady state, the impedance Z_{eq} seen by the utility source is

$$Z_{eq} = R_s + j\omega_0 L_{ls} + (j\omega_0 L_m) \| (j\omega_0 L_{lr} + R_r + R_x)$$

Plugging in the component values, we find that

$$Z_{eq} = 5.72 + 12.82\, j$$

With the relation $I = V / Z_{eq}$ we have

$$I = \frac{V}{Z_{eq}} = \frac{170}{5.72 + 12.82\, j} e^{j\omega_0 t} = 12.1\, e^{-1.15 j}\, e^{j\omega_0 t}$$

Since $i(t) = \Re\{I\}$, $i(t) = 12.1 \cos(377t - 1.15)$. This answers the first part of the question.

The time average power is $\langle P \rangle = \frac{1}{2} \Re\{V_o \bar{I}_o\}$ where $\bar{I}_o$ denotes the complex conjugate of I_o . This relation gives

$$\langle P \rangle = \frac{1}{2} \Re\{170 * 12.1 e^{1.15 j}\} = 420\, W$$

and this answers the second part of the question.