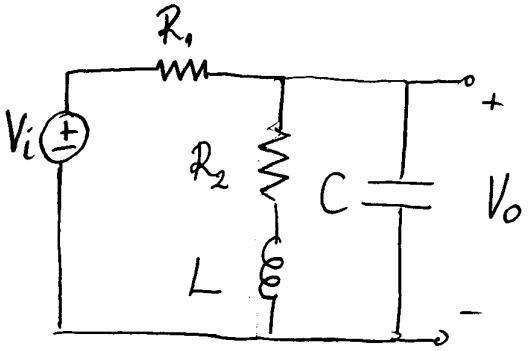
PROBLEM SET 9 SOLUTIONS

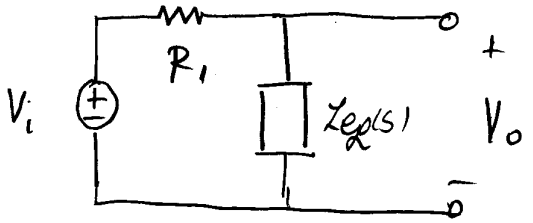
6.002, FALL '01



$R_1 = 1k\Omega$
$R_2 = 250\Omega$
$L = 50mH$
$C = 1\mu F$

a) $H(s) = ?$

Simlified network:



$$Z_{eq} = (R_2 + Ls) \,||\, \frac{1}{Cs}$$

$$= \frac{(R_2 + Ls)\cdot \frac{1}{Cs}}{R_2 + Ls + \frac{1}{Cs}}$$

$$= \frac{R_2 + Ls}{LCs^2 + R_2 Cs + 1}$$

From the divider relationship:

$$V_o = \frac{Z_{eq}\cdot V_i}{R_1 + Z_{eq}} \Rightarrow H(s) = \frac{V_o}{V_i} = \frac{Z_{eq}}{R_1 + Z_{eq}}$$

$(LCs^2 + R_2Cs + 1)$

$$\Rightarrow H(s) = \frac{\frac{R_2 + Ls}{LCs^2 + R_2Cs + 1}}{R_1 + \frac{R_2 + Ls}{LCs^2 + R_2Cs + 1}} = \frac{R_2 + Ls}{R_1LCs^2 + R_1R_2Cs + R_1 + R_2 + Ls}$$

$$H(s) = \frac{R_2 + Ls}{R_1 LC s^2 + (R_1 R_2 C + L)s + R_1 + R_2}$$

$$\Rightarrow \boxed{H(s) = \frac{250 + 0.05s}{5 \cdot 10^{-5} s^2 + 0.3s + 1250}}$$

b)

to find zeros:

$$250 + 0.05s = 0$$

$$\Rightarrow \boxed{s_z = 5000}$$

to find poles:

$$5 \cdot 10^{-5} s^2 + 0.3s + 1250 = 0$$

$$s_{p1,2} = \frac{-0.3 \pm \sqrt{(0.3)^2 - 4 \cdot 5 \cdot 10^{-5} \cdot 1250}}{2 \cdot 5 \cdot 10^{-5}}$$

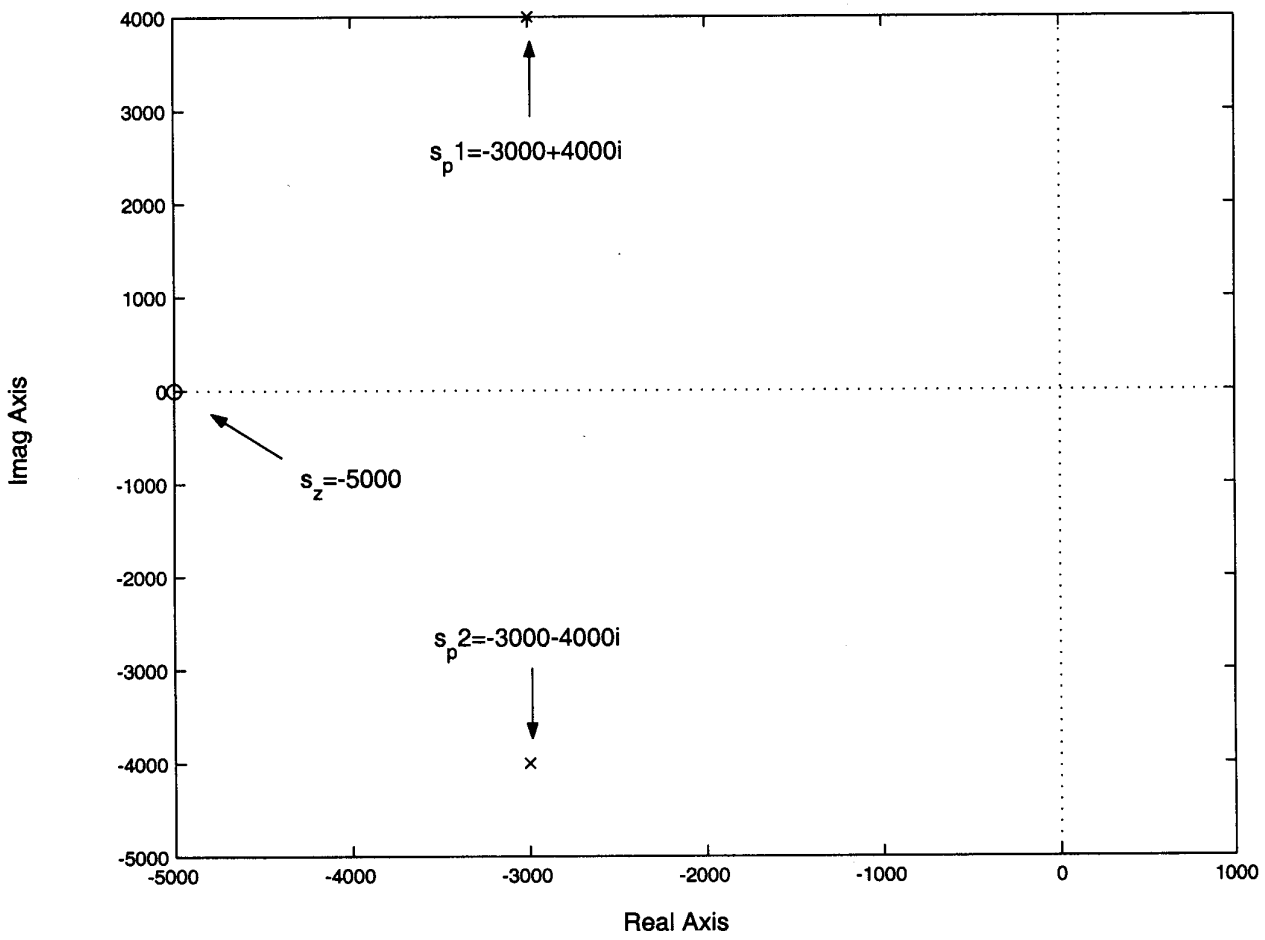
$$\Rightarrow \boxed{\begin{aligned} s_{p1} &= -3000 + 4000j \\ s_{p2} &= -3000 - 4000j \end{aligned}}$$

→ complex solns come in conjugate pairs

$$\Rightarrow H(s) = K \frac{(s - s_z)}{(s - s_{p1})(s - s_{p2})}, \quad K = \frac{0.05}{5 \cdot 10^{-5}} = \frac{5 \cdot 10^{-2}}{5 \cdot 10^{-5}} = \boxed{1000 = K}$$

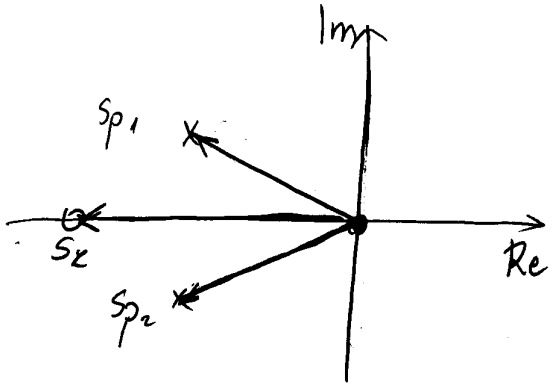
$$|H(s)| = K \frac{|s - s_z|}{|s - s_{p1}|\,|s - s_{p2}|}, \quad s = j\omega$$

Problem 9.1: Pole-Zero Plot in the Complex s-plane



From the last equation on page 2 we see that the magnitude of the transfer function $|H(j\omega)|$ is equal to K times the ratio of the distance from zero to $Im(j\omega, s)$ axis and the product of the distances from poles to Im axis.

1) When $\omega \to 0$



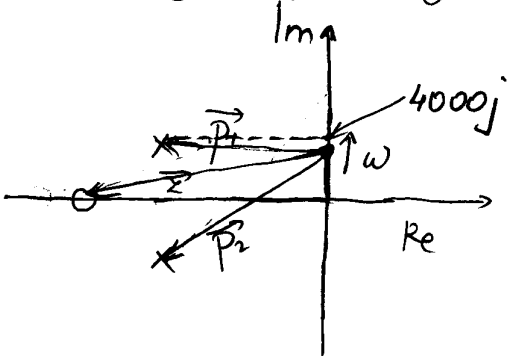
$$|H(j\cdot 0)| = K \cdot \frac{|S_z|}{|Sp_1|\,|Sp_2|} =$$

$$= 1000 \cdot \frac{5000}{\sqrt{3000^2+4000^2}\,\sqrt{3000^2+4000^2}}$$

$$= \frac{5\cdot 10^6}{3000^2+4000^2} = \frac{5 \cdot 10^6}{(9+16)\cdot 10^6}$$

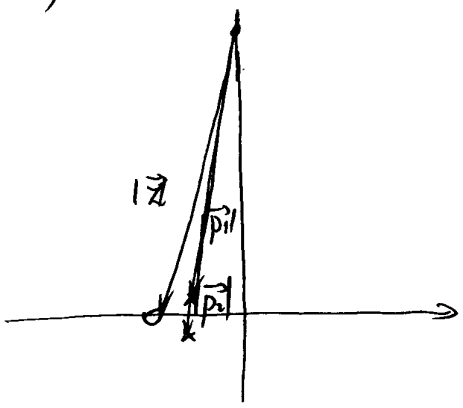
$$= \boxed{\frac{1}{5} = |H(j\cdot 0)|}$$

2) as we increase ω $|\vec{z}|$ increases, $|\vec{p}_2|$ increases too, however, $|\vec{p}_1|$ decreases



Overall, the increase in $|\vec{p}_2|$ will cancel the effect of the increase in $|\vec{z}|$ so we will see a slight increase in $|H(j\omega)|$. This happens approx. until $\omega = 4000$. Post this frequency $|\vec{p}_1|$ will increase too $\Rightarrow |H(j\omega)|$ will decrease

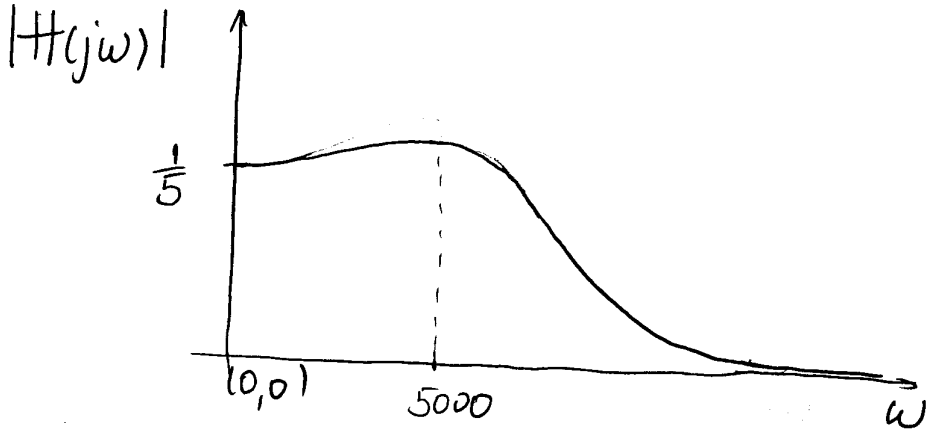
3) as $\omega \to 0$ $\quad |\vec{z}| \approx |\vec{p_1}| \approx |\vec{p_2}|$



$$\Rightarrow H(j\omega) \approx K \cdot \frac{|\vec{z}|}{|\vec{p_1}|\,|\vec{p_2}|} = \frac{K}{|j\omega - s_{p_2}|} \quad (|j\omega - s_{p_2}| \to \infty)$$

$$H(j\infty) = \frac{K}{\infty} = 0$$

$\Rightarrow$

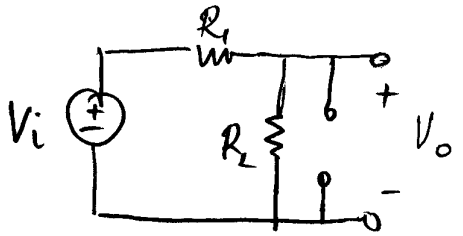


c) $s \to 0 \Rightarrow \omega \to 0$ (DC)

$Z_c = \frac{1}{Cs} \xrightarrow{s \to 0} \infty \Rightarrow$ capacitor is OPEN

$Z_L = L \cdot s \xrightarrow{s \to 0} 0 \Rightarrow$ Inductor is SHORT

$Z_R = R \xrightarrow{s \to 0} R$

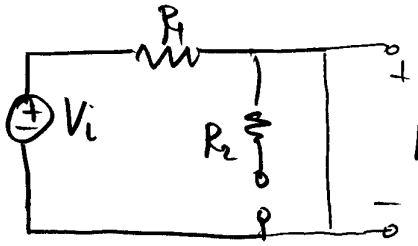


$$\Rightarrow V_o = \frac{R_2}{R_1 + R_2} V_i$$

$$\Rightarrow |H(s)| = \frac{R_2}{R_1 + R_2} = \frac{250}{1250} = \boxed{\frac{1}{5} = |H(0)|}$$

↑ agrees w/ a) & b)

$S \to \infty$ $\begin{cases} Z_C \to 0 \Rightarrow \text{capacitor is SHORT} \\ Z_L \to \infty \Rightarrow \text{inductor is OPEN} \\ Z_R \to R \end{cases}$

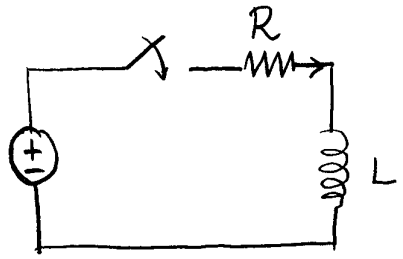


$V_o \Rightarrow V_o = 0$

$$\frac{V_o}{V_i} = \boxed{H(\infty) = 0}$$

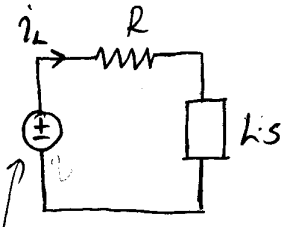
↑ agrees w/ a) & b)

(2)



$v_i(t) = V_i \cos(\omega t + \phi)$

a)



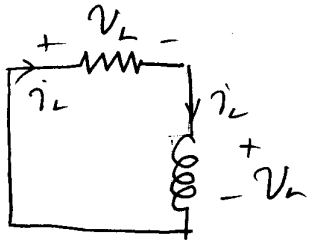
$t > 0$

↑ suddenly applied cosine wave → TONE BURST

$i_L(t) = i_{L_H}(t) + i_{L_P}(t)$

↑ HOMOGENEOUS SOLUTION ↑ PARTICULAR SOLUTION

to find the homogeneous soln:



$$i_L \cdot R + v_L = 0$$

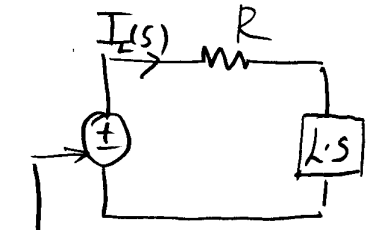
$$i_L \cdot R + L\frac{di_L}{dt} = 0$$

$$i_L = k_1 e^{s_1 t}$$

$$\Rightarrow \quad R + L \cdot s_1 = 0$$

$$\Rightarrow \quad s_1 = -\frac{R}{L}$$, k_1 is to be found from initial conditions

Particular or sinusoidal steady-state (SSS) solution can be found using the impedance method



$$V_i(s) = \mathrm{Im}\{V_i e^{j(\omega t+\phi)}\}$$

$$I_L(s) = \frac{V_i(s)}{R + L\cdot s} = \frac{V_i e^{j(\omega t+\phi)}}{\sqrt{R^2+L^2\omega^2}\cdot e^{j\arctan\frac{L\omega}{R}}}$$

$R + L\cdot s$ ⇝ $R + j\omega L$

$$\Rightarrow \quad I_L(s) = \frac{V_i}{\sqrt{R^2+L^2\omega^2}} e^{j(\omega t+\phi-\arctan\frac{L\omega}{R})}$$

$$i_{Lp}(t) = \mathrm{Im}\{I_L(s)\}$$

$$i_{Lp}(t) = \frac{V_i}{\sqrt{R^2+L^2\omega^2}} \sin\left(\omega t+\phi-\arctan\frac{L\omega}{R}\right)$$

$$\Rightarrow i_{L(t)_{TOTAL}} = K_1 e^{\frac{-Rt}{L}} + \frac{V_i}{\sqrt{R^2+L^2\omega^2}} \sin(\omega t + \phi - \arctan\frac{L\omega}{R}) \quad , t>0$$

to find K_1:

$$i_L(0) = K_1 + \frac{V_i}{\sqrt{R^2+L^2\omega^2}} \sin(\phi - \arctan\frac{L\omega}{R}) = 0$$

let $\phi_1 = \arctan\frac{L\omega}{R}$

$$\Rightarrow K_1 = -\frac{V_i}{\sqrt{R^2+L^2\omega^2}} \sin(\phi - \phi_1)$$

(equation x)

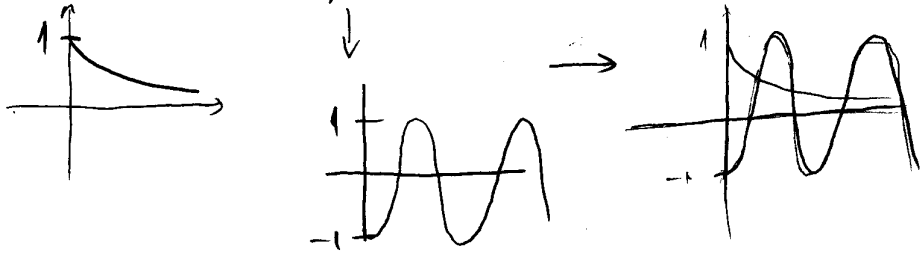
EQ X

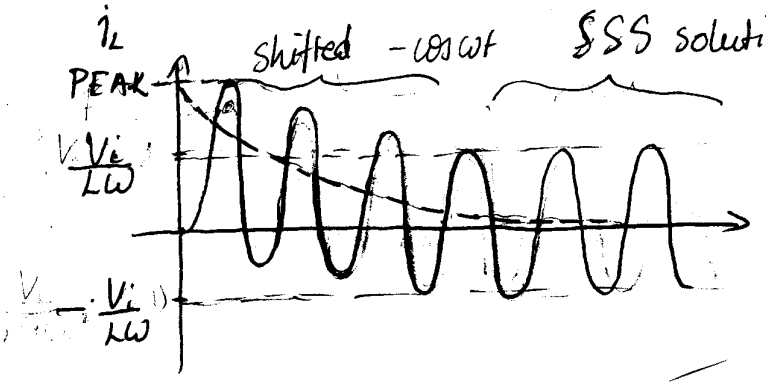
$$\Rightarrow \boxed{i_{L(t)_{TOTAL}} = -\frac{V_i}{\sqrt{R^2+L^2\omega^2}} \sin(\phi-\phi_1) \cdot e^{\frac{-R\cdot t}{L}} + \frac{V_i}{\sqrt{R^2+L^2\omega^2}} \sin(\omega t + \phi - \phi_1)}$$

b) $\phi = 0$, $\frac{\omega L}{R} \gg 1 \quad (\to \infty) \Rightarrow \phi_1 = \frac{\pi}{2}$

$$\Rightarrow i_{L(t)_{TOTAL}} = +\frac{V_i}{\sqrt{R^2+(L\omega)^2}} \sin(+\tfrac{\pi}{2}) e^{\frac{-R\cdot t}{L}} + \frac{V_i}{\sqrt{R^2+L^2\omega^2}} \underbrace{\sin(\omega t - \tfrac{\pi}{2})}_{-\cos(\omega t)}$$

$$\Rightarrow i_{L(t)_{TOTAL}} = \frac{V_i}{\sqrt{R^2+(L\omega)^2}} \left[e^{\frac{-R\cdot t}{L}} - \cos\omega t \right]$$





the peak value occurs when $\omega t = \pi \Rightarrow t = \frac{\pi}{\omega}$

$$\Rightarrow i_{L\max} = \frac{V_i}{\sqrt{R^2+(L\omega)^2}}\left[e^{-\frac{R}{L}\frac{\pi}{\omega}} + 1\right] = \frac{V_i}{L\omega}\left[e^{-0}+1\right] = \frac{2V_i}{L\omega}$$

$$i_{L_{SSS}} = i_{L_P}\Big|_{\phi=0,\ \frac{\omega L}{R}\gg 1} = \frac{V_i}{\sqrt{R^2+(L\omega)^2}}\sin\left(\omega t - \frac{\pi}{2}\right) = \frac{V_i}{L\omega}\sin\left(\omega t - \frac{\pi}{2}\right)$$

($\sqrt{R^2+(L\omega)^2} \approx L\omega$ since $\frac{\omega L}{R}\gg 1$)

in SSS amplitude $= \frac{V_i}{L\omega}$

$$\Rightarrow \boxed{i_{L\max} = \frac{2V_i}{L\omega}}$$

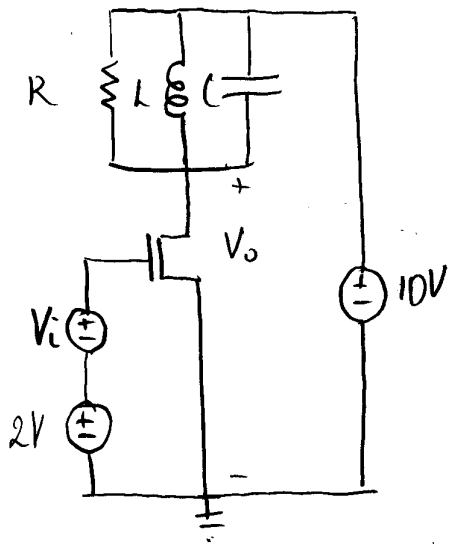
c) From EQ x we see that particular solution is eliminated when $\phi = \phi_1$ since $\sin(\phi-\phi_1) = 0$ in that case

$$\Rightarrow \boxed{\phi = \arctan\frac{L\omega}{R}}$$

$i_L(t)$ now becomes

$$\boxed{i_L(t) = \frac{V_i}{\sqrt{R^2+L^2\omega^2}}\sin\omega t}, \quad t>0$$

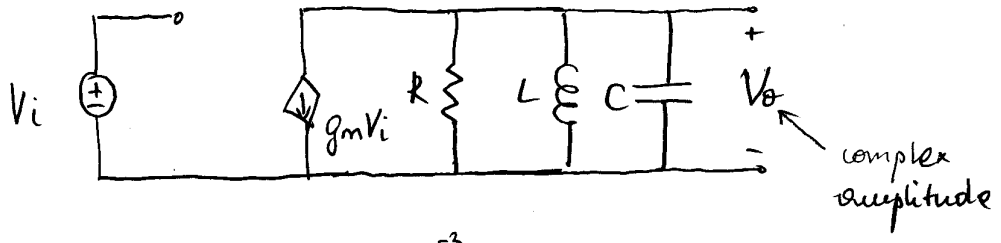
(3)



$v_i(t) = V_i \cos \omega t$

$R = 5\sqrt{\frac{L}{C}}$

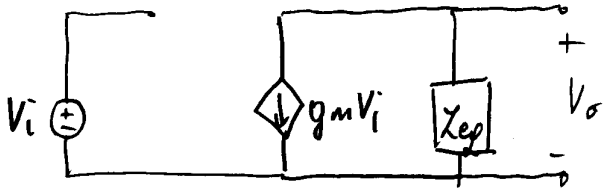
a)



$g_m = K(V_{GS} - V_T) = 2 \cdot 10^{-3} \cdot (2-1)$

$\boxed{g_m = 2mS}$

simplified circuit:



$Z_{eq} = R \,\|\, Ls \,\|\, \frac{1}{Cs}$

$= \frac{1}{\frac{1}{R} + \frac{1}{Ls} + Cs}$

$= \boxed{\frac{1}{\frac{1}{R} + j\left(C\omega - \frac{1}{L\omega}\right)} = Z_{eq}}$

b) $H(s) = \frac{V_o}{V_i} = -g_m Z_{eq} = \boxed{-\frac{2 \cdot 10^{-3}}{\frac{1}{R} + j\left(C\omega - \frac{1}{L\omega}\right)} = H(j\omega)}$

$\boxed{H(s) = -\frac{2 \cdot 10^{-3}}{\frac{1}{R} + \frac{1}{Ls} + Cs}}$

c) $|H(j\omega)| = \left| \frac{-2\cdot 10^{-3}}{R + j(C\omega - \frac{1}{L\omega})} \right| = \frac{2\cdot 10^{-3}}{\sqrt{(R)^2 + (C\omega - \frac{1}{L\omega})^2}}$

$\omega \to 0$

$|H(j\cdot 0)| = \frac{2\cdot 10^{-3}}{\sqrt{(R)^2 + \infty}} = 0$

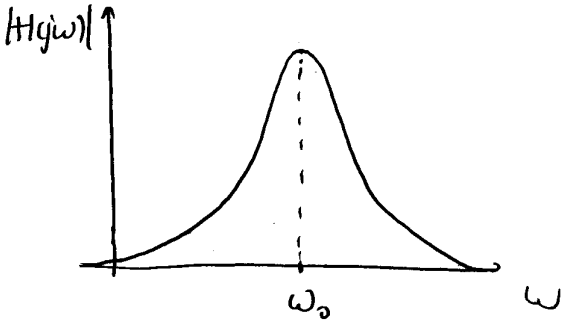
$\omega \to \infty$

$|H(j\infty)| = 0$

} bandpass filter characteristics

$|H(j\omega)|_{max} = \frac{2\cdot 10^{-3}}{\sqrt{(R)^2 + 0}} = 2\cdot 10^{-3} R = 2\cdot 10^{-3}\cdot 5\sqrt{\frac{L}{C}}$

$\boxed{|H(j\omega)|_{max} = 10^{-2}\sqrt{\frac{L}{C}}}$, $\omega_0 = \frac{1}{\sqrt{LC}}$



$\arg H(j\omega) = \arg(\text{numerator}) - \arg(\text{denominator}) = \arg(\underbrace{-1}_{e^{j\pi}}) - \tan^{-1}\left(R(C\omega - \frac{1}{L\omega})\right)$

$\arg H(j\omega) = \pi - \tan^{-1}\left(\underbrace{R}_{5\sqrt{\frac{L}{C}}} C\omega - \underbrace{\frac{R}{L}}_{5\sqrt{\frac{L}{C}}} \frac{1}{\omega}\right) = \boxed{\pi - \tan^{-1}\left(5\sqrt{LC}\,\omega - \frac{5}{5\sqrt{LC}}\frac{1}{\omega}\right)}$

$\omega \to 0$

$\arg H(j\omega) \to \pi - \tan^{-1}(-\infty) = \pi + \underbrace{\tan^{-1}(\infty)}_{\frac{\pi}{2}} = \frac{3\pi}{2} \sim 4.71$

$\boxed{\omega \to \omega_0 = \frac{1}{\sqrt{LC}}}$

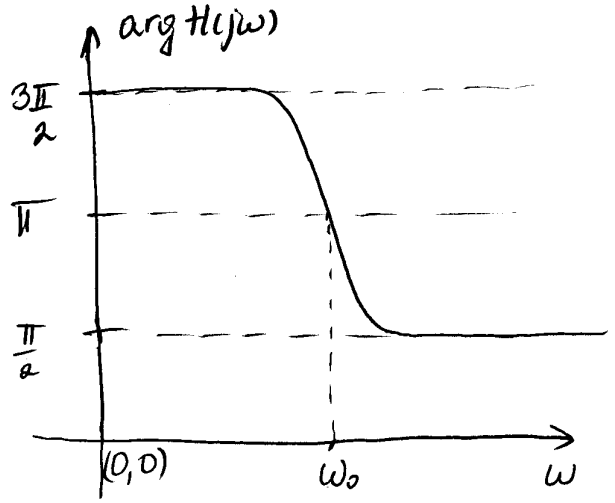
$\arg H(j\omega_0) = \pi - \tan^{-1}(5 - 5) = \pi - \tan^{-1}(0) = \pi - 0 = \pi$

$\tan^{-1}$: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\omega \to \infty$

$\arg H(j\infty) = \pi - \tan^{-1}(\infty) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

$\Rightarrow$



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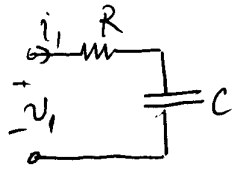
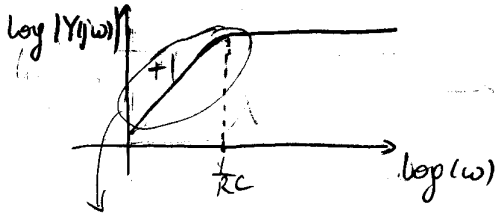


$$Y(j\omega) = \frac{I_1(j\omega)}{V_1(j\omega)} = \frac{1}{R + \frac{1}{Cs}} = \frac{Cs}{RCs+1} = \frac{j\omega C}{j\omega RC + 1}$$

zero: $z_1 = 0$

pole: $p = -\frac{1}{RC}$



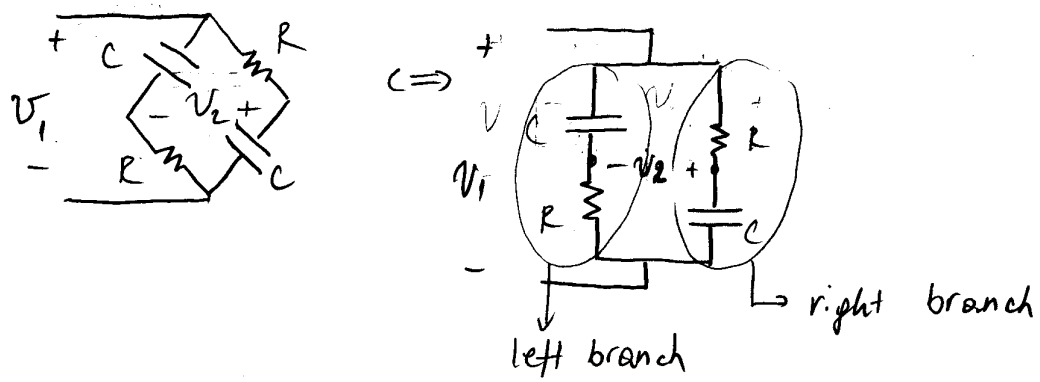
(*) at very low frequencies $j\omega RC$ term $\ll 1$

$\Rightarrow Y(j\omega) \sim \omega \Rightarrow \log|Y(j\omega)| \sim \log\omega \to$ slope $= +1$

(*) at high frequencies $j\omega RC \gg 1$

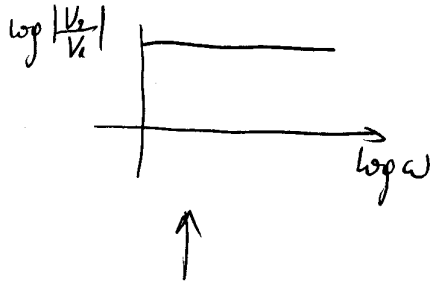
$\Rightarrow Y(j\omega) = \frac{j\omega C}{j\omega RC} = \frac{1}{R} \to$ constant

$\Rightarrow$ The plot of the magnitude of $Y(j\omega)$ corresponds to figure (2)

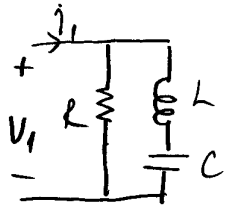


because left and right branches have the same impedence, they will have the same current flowing through them. Thus, the voltage drop across the resistors in both branches will be the same and voltage drops across the capacitors in both branches will be the same

$\Rightarrow V_1 = V_2 \Rightarrow \frac{V_2(j\omega)}{V_i(j\omega)} = 1$



looks like figure 4



$$Z(j\omega) = R \parallel \left(Ls + \frac{1}{Cs}\right) = \frac{R(LCs^2+1)}{LCs^2+RCs+1} = \frac{R(1-LC\omega^2)}{j\omega RC + (1-LC\omega^2)}$$

$$\boxed{|Z(j\omega)| = \frac{R(1-LC\omega^2)}{\sqrt{(\omega RC)^2 + (1-LC\omega^2)^2}}}$$

$\omega \to 0$

$\boxed{|Z(j\cdot 0)| = R}$ → constant

$\omega \to \infty$

$\boxed{|Z(j\omega)| = \frac{R\cdot(-LC\omega^2)}{-LC\omega^2} = R}$ → constant

$\omega_c \to \frac{1}{\sqrt{LC}}$

$\boxed{|Z(j\frac{1}{\sqrt{LC}})| = 0}$

NOTCH FILTER CHARACTERISTICS

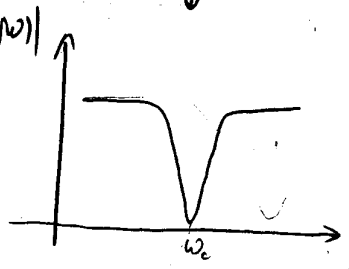
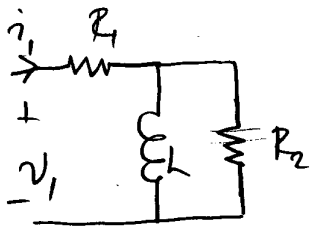


figure 8



$$Z(j\omega) = R_1 + (R_2 \| Ls) = R_1 + \frac{R_2 Ls}{Ls + R_2}$$

$\omega \to 0$

$Z(0) \to \boxed{R_1}$ $\quad \log(Z(0)) = \log R_1 \to$ constant

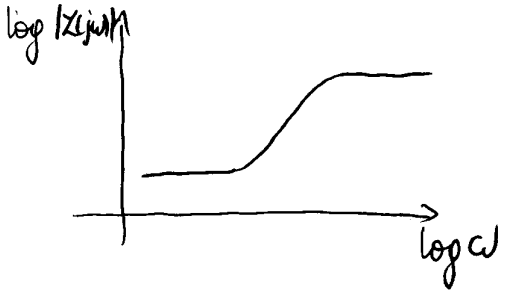
$\omega \to \infty$

$Z(\infty) = R_1 + \frac{R_2 Ls}{Ls} = \boxed{R_1 + R_2}$ (inductor is open)

$\log|Z(\infty)| = \log(R_1 + R_2) \to$ constant $> \log R_1$

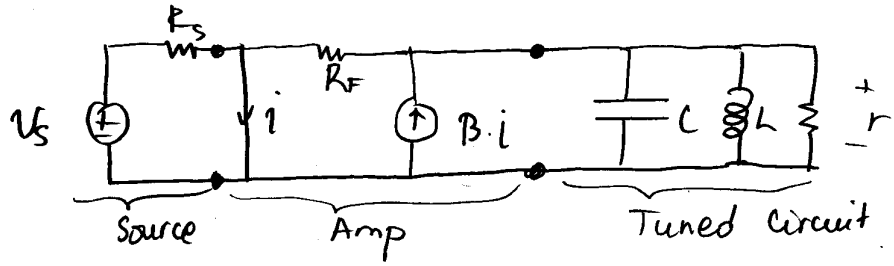
Somewhere between $\omega = 0$ & $\omega = \infty$ we must have a transitional region in which $Z(j\omega)$ increases from R_1 to $R_1 + R_2$

$\Rightarrow$



$\to$ the only figure that looks like this is figure (5)

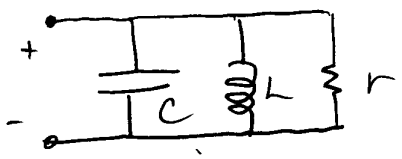
(5)



$R_S = 1k\Omega$ $\quad L = \frac{100}{11}\mu H$

$r = 10k\Omega$ $\quad \beta = 11$

a)



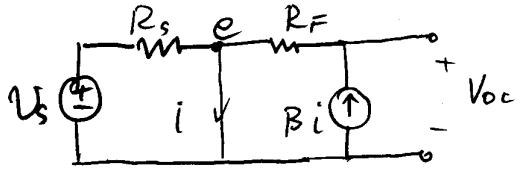
Q = quality factor = $\frac{\text{center frequency}}{\text{bandwidth}}$

for parallel RLC circuit

$Q = \frac{r}{\omega_0 L} = 50$

6) $H(s) = \frac{V_o}{V_s} = ?$

step 1: find the Thevenin equivalent of the source & Amp



$V_{oc} = R_F \cdot \beta \cdot i$

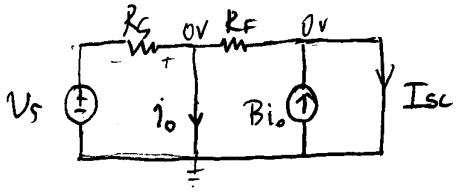
to find i in terms of V_s:

KCL @ e

$\frac{V_s}{R_s} + \beta \cdot i = i$

$\frac{V_s}{R_s} = (-\beta + 1) \cdot i$

$\frac{V_s}{R_s} = -10\, i \Rightarrow \boxed{i = \frac{1}{(1-\beta)R_s} V_s}$



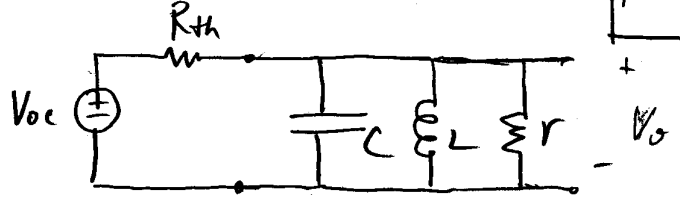
$i_o = + \frac{V_s}{R_s}$

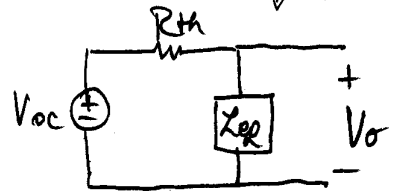
$I_{sc} = \beta \cdot i_o \Rightarrow$

$\boxed{V_{oc} = \frac{R_F \cdot \beta}{(1-\beta)R_s} \cdot V_s}$

$R_{th} = \frac{V_{oc}}{I_{sc}} = \frac{R_F \cdot \beta \cdot i}{\beta \cdot i_o} = \frac{R_F \frac{1}{(1-\beta)R_s} \cdot V_s}{+\frac{V_s}{R_s} \cdot \beta}$

$\boxed{R_{th} = \frac{R_F}{1-\beta}}$





$Z_{eq} = \frac{1}{Cs} \parallel Ls \parallel r = \frac{1}{Cs + \frac{1}{Ls} + \frac{1}{r}}$

$Z_{eq} = \frac{rLs}{rLCs^2 + Ls + r}$

$V_o = \frac{Z_{eq}}{R_{th} + Z_{eq}} \cdot V_{oc} = \frac{\frac{rLs}{rLCs^2 + Ls + r}}{\frac{rLs}{rLCs^2 + Ls + r} + R_{th}} \cdot V_{oc}$

$$V_o = \frac{r_L s}{R_{th} \cdot r_L C s^2 + (R_{th} L + r_L) s + R_{th} \cdot r} \cdot \frac{R_F \cdot \beta}{(1-\beta) \cdot R_s} \cdot V_s$$

$$\Rightarrow H(s) = \frac{V_o}{V_s} = \boxed{\frac{R_F \cdot \beta}{(1-\beta) \cdot R_s} \cdot \frac{r_L s}{R_{th} \cdot r_L C s^2 + (R_{th} L + r_L) s + R_{th} \cdot r} = H(s)}$$

c) $C = ?$, $R_F = ?$ $\quad Q = ?$ $\quad Q = \frac{\omega_0}{2\alpha} = \frac{10^6}{2 \cdot 2 \cdot 10^3} = \boxed{250 = Q}$

$\omega_0 = 1\,MHz \cdot 2\pi$

$2\alpha = 2\,kHz \cdot 2\pi$

$$H(j\omega) = \text{constant} \cdot \frac{j\omega}{(R_{th} L + r_L) \cdot j\omega + (R_{th} \cdot r - R_{th} r_L C \omega^2)}$$

$\Rightarrow$ $H(j\omega)$ is max when ~~$R_{th} \cdot r$~~ $- R_{th} r_L C \omega^2 = 0$

$$\Rightarrow \omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow C = \frac{1}{\omega_0^2 \cdot L} = \boxed{\frac{10}{4\pi}\,nF = C} = 0.7958\,nF$$

from the characteristic equation

$R_{th} \cdot r_L C s^2 + (R_{th} L + r_L) s + R_{th} \cdot r = 0 \;/ : R_{th} r_L C$

$$s^2 + \frac{R_{th} \cdot L + r L}{R_{th} \cdot r L C} s + \frac{1}{LC} = s^2 + 2\alpha s + \omega_0^2 = 0$$

$$\Rightarrow \alpha = \frac{1}{2} \cdot \frac{R_{th} + r}{R_{th} \cdot r C}$$

$$R_{th} = \frac{R_F}{1-\beta} = \frac{R_F}{-10}$$

$\Rightarrow R_F = -10 R_{th}$

$\Rightarrow \boxed{R_F = 111.1\,k\Omega}$

$2\alpha R_{th} \cdot r \cdot C = R_{th} + r$

$R_{th}(2\alpha r \cdot C - 1) = r \Rightarrow R_{th} = \frac{r}{2\alpha r C - 1} = 11.1\,k\Omega$