6.003: Signals and Systems

Feedback and Control

September 24, 2009

Last Time

Understanding the structure of a control problem

automatic control → feedback

Analyzing feedback systems

feedback → cyclic paths → persistent outputs

Designing control systems

constructing well-behaved response properties

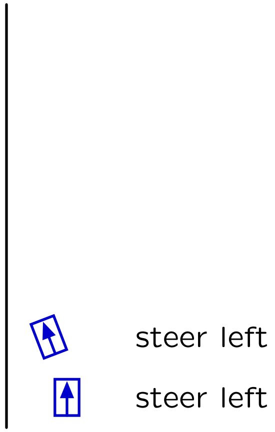
Example: Steering a Car

Algorithm: steer left when car is right of center and vice versa.

steer left

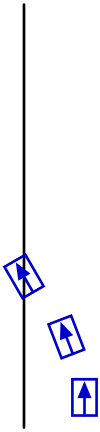
Example: Steering a Car

Algorithm: steer left when car is right of center and vice versa.



Example: Steering a Car

Algorithm: steer left when car is right of center and vice versa.



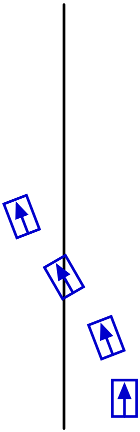
straight ahead?

steer left

steer left

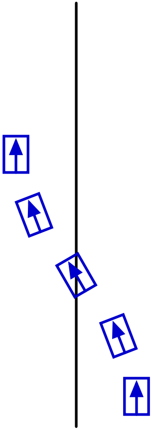
Example: Steering a Car

Algorithm: steer left when car is right of center and vice versa.



Example: Steering a Car

Algorithm: steer left when car is right of center and vice versa.



steer right

steer right

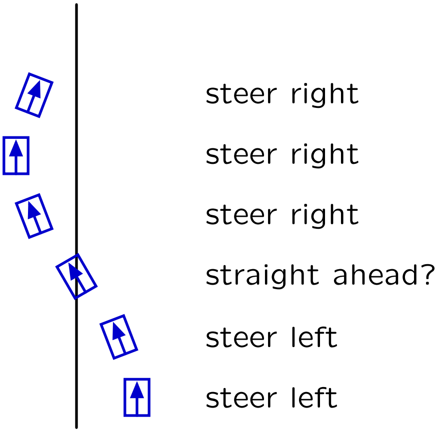
straight ahead?

steer left

steer left

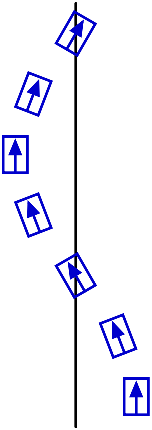
Example: Steering a Car

Algorithm: steer left when car is right of center and vice versa.



Example: Steering a Car

Algorithm: steer left when car is right of center and vice versa.



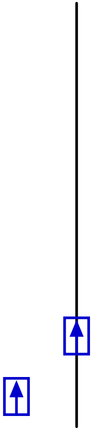
Bad algorithm $\rightarrow$ poor performance.

Here we get persistent oscillations!

Last Time

We investigated a VERY simple model for the car.

The car could move laterally in a lane without rotating!

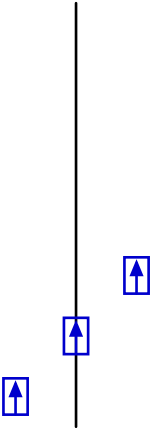


see previous $(-1) \rightarrow$ command 1

Last Time

We investigated a VERY simple model for the car.

The car could move laterally in a lane without rotating!



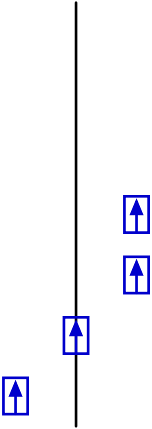
see previous $(0) \rightarrow$ command 0

see previous $(-1) \rightarrow$ command 1

Last Time

We investigated a VERY simple model for the car.

The car could move laterally in a lane without rotating!



see previous $(+1) \rightarrow$ command -1

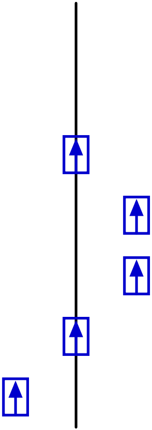
see previous $(0) \rightarrow$ command 0

see previous $(-1) \rightarrow$ command 1

Last Time

We investigated a VERY simple model for the car.

The car could move laterally in a lane without rotating!



see previous $(+1) \rightarrow$ command -1

see previous $(+1) \rightarrow$ command -1

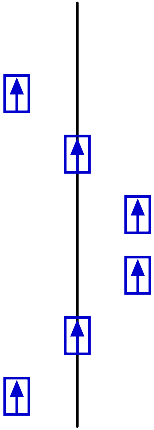
see previous $(0) \rightarrow$ command 0

see previous $(-1) \rightarrow$ command 1

Last Time

We investigated a VERY simple model for the car.

The car could move laterally in a lane without rotating!



see previous $(0) \rightarrow$ command 0

see previous $(+1) \rightarrow$ command -1

see previous $(+1) \rightarrow$ command -1

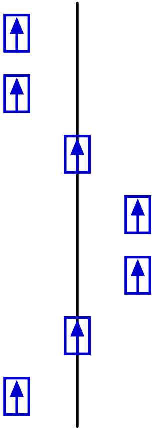
see previous $(0) \rightarrow$ command 0

see previous $(-1) \rightarrow$ command 1

Last Time

We investigated a VERY simple model for the car.

The car could move laterally in a lane without rotating!



see previous $(0) \rightarrow$ command 0

see previous $(+1) \rightarrow$ command -1

see previous $(+1) \rightarrow$ command -1

see previous $(0) \rightarrow$ command 0

see previous $(-1) \rightarrow$ command 1

Even that simple system could go unstable.

Last time, we learned how to stablize it.

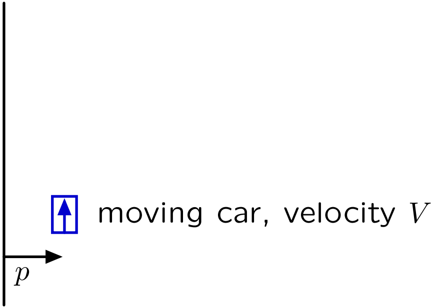
Today

We will investigate more realistic models of steering.

We will analyze more realistic (and more complex) feedback systems.

Steering Controller

Design a system to automatically steer a car.



Assume a sensor reports position p within the lane:

$p = 0$: in center

$p > 0$: right of center

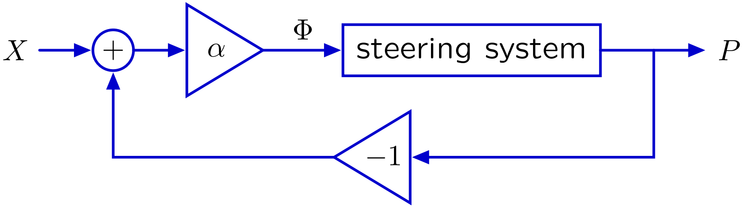
$p < 0$: left of center

Steering Controller

Model the system.

Let X represent the desired position in the lane (normally 0).

Turn the steering wheel (ϕ) in proportion to the difference between the desired and current positions.

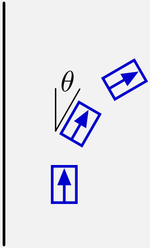


The relation between the angle of the steering wheel and the position of the car in the lane is complicated:

- turning the steering wheel causes the car to rotate.
- forward motion of the car then alters its position in the lane.

Check Yourself

What is the (CT) relation between ϕ (angle of steering wheel) and θ (angle of car)?

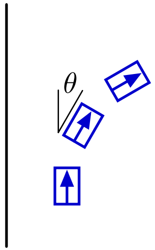


1. $\theta \propto \phi$
2. $\theta \propto \sin\phi$
3. $\dot{\theta} \propto \phi$
4. $\theta \propto \dot{\phi}$
5. none of the above

Check Yourself

What is the (CT) relation between ϕ (angle of steering wheel) and θ (angle of car)?

$\phi =$ constant $\rightarrow$ circles



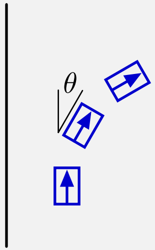
circles $\rightarrow \theta$ increases at a constant rate

The rate is proportional to ϕ.

$$\dot{\theta} \propto \phi$$

Check Yourself

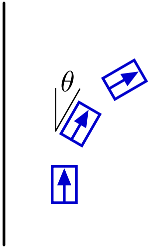
What is the (CT) relation between ϕ (angle of steering wheel) and θ (angle of car)? 3



1. $\theta \propto \phi$
2. $\theta \propto \sin\phi$
3. $\dot{\theta} \propto \phi$
4. $\theta \propto \dot{\phi}$
5. none of the above

Steering Controller

Make a DT model.

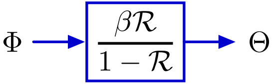


$$\dot{\theta} \propto \phi$$

$$\frac{\theta[n+1] - \theta[n]}{\Delta} \propto \phi[n]$$

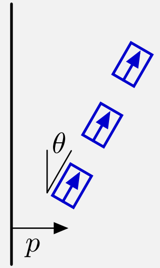
$$\theta[n+1] = \theta[n] + \beta\phi[n] \;; \qquad \beta = \Delta \times \text{proportionality constant}$$

$$\Theta = \mathcal{R}\Theta + \beta\mathcal{R}\Phi$$



Check Yourself

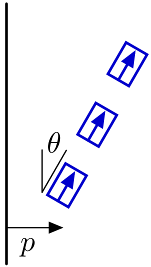
What is the (CT) relation between θ (angle of car) and p (position of car in lane)?



1. $\dot{p} \propto \theta$
2. $\dot{p} \propto \sin\theta$
3. $p \propto \dot{\theta}$
4. $p \propto \frac{d\sin\theta}{dt}$
5. none of the above

Check Yourself

What is the (CT) relation between θ (angle of car) and p (position of car in lane)?



$\theta =$ constant $\rightarrow$ linear increase in p

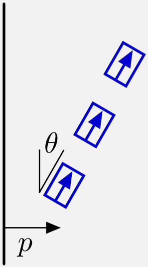
Change in p proportional to product of V and $\sin\theta$

$$\dot{p} = V \sin\theta$$

where V is forward velocity of the car.

Check Yourself

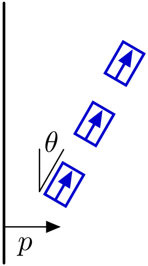
What is the (CT) relation between θ (angle of car) and p (position of car in lane)? 2



1. $\dot{p} \propto \theta$
2. $\dot{p} \propto \sin\theta$
3. $p \propto \dot{\theta}$
4. $p \propto \frac{d \sin\theta}{dt}$
5. none of the above

Steering Controller

Make a DT model.

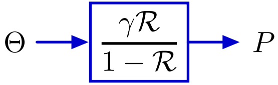


$$\dot{p} = V \sin\theta$$

$$\frac{p[n+1]-p[n]}{\Delta} \propto V\theta[n] \quad \text{(small angle approximation)}$$

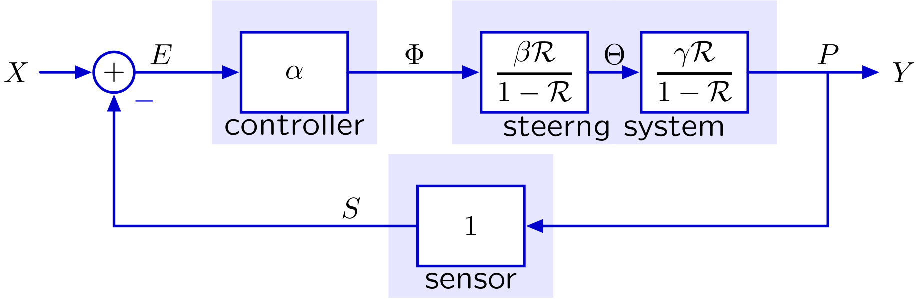
$$p[n+1] = p[n] + \gamma\theta[n] \;; \quad \gamma = V\Delta \times \text{proportionality constant}$$

$$P \approx \mathcal{R}P + \gamma\mathcal{R}\Theta$$



Steering Controller

Combining the relations for Φ, Θ, and P provides the following model.



Determine the system functional.

$$Y = \alpha \frac{\beta\mathcal{R}}{1-\mathcal{R}} \; \frac{\gamma\mathcal{R}}{1-\mathcal{R}} (X - Y)$$

Solving:

$$\frac{Y}{X} = \frac{\alpha\beta\gamma\mathcal{R}^2}{1 - 2\mathcal{R} + (1+\alpha\beta\gamma)\mathcal{R}^2} = \frac{K\mathcal{R}^2}{1 - 2\mathcal{R} + (1+K)\mathcal{R}^2}$$

where $K \equiv \alpha\beta\gamma$.

Steering Controller

To find the poles, replace $\mathcal{R}$ in the system functional by $\frac{1}{z}$ and solve for the roots of the denominator.

$$\frac{Y}{X} = \frac{K\mathcal{R}^2}{1 - 2\mathcal{R} + (1+K)\mathcal{R}^2} = \frac{K}{z^2 - 2z + (1+K)}$$

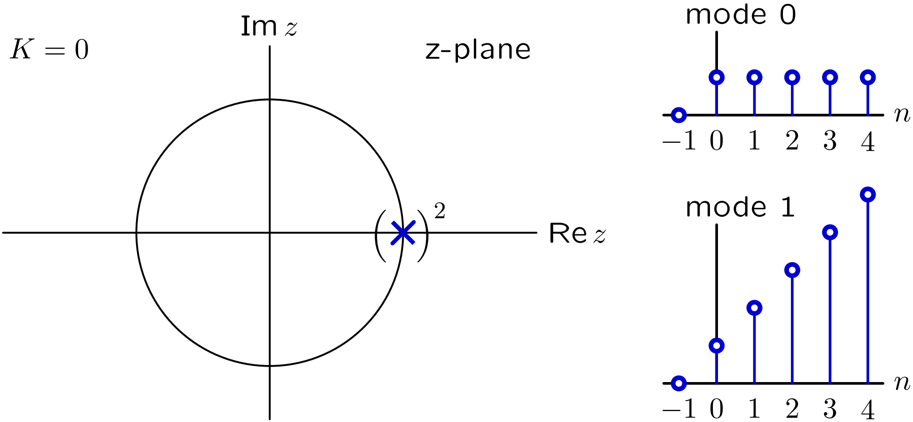
Poles are at

$$z = 1 \pm j\sqrt{K}\,.$$

Steering Controller

If $K = 0$, there is a double pole at $z = 1$.

$$z = 1 \pm j\sqrt{K}$$

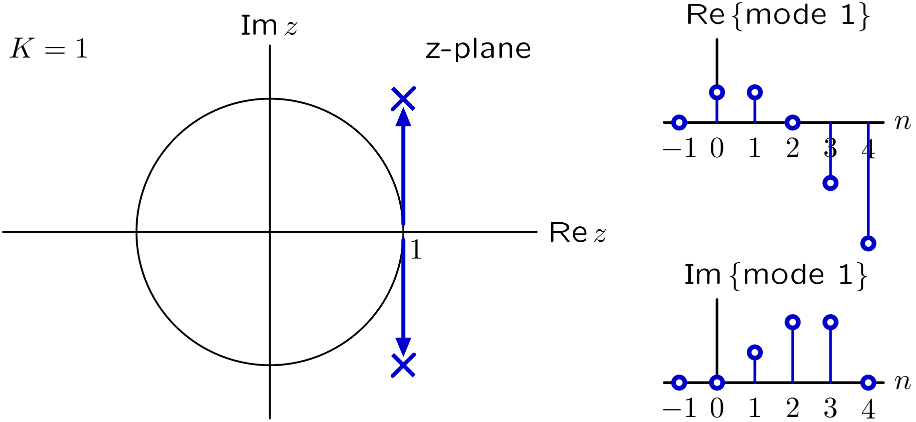


The output diverges: the system is "unstable."

Steering Controller

If $K = 1$, there are complex poles at $z = 1 \pm j$.

$$z = 1 \pm j\sqrt{K}$$



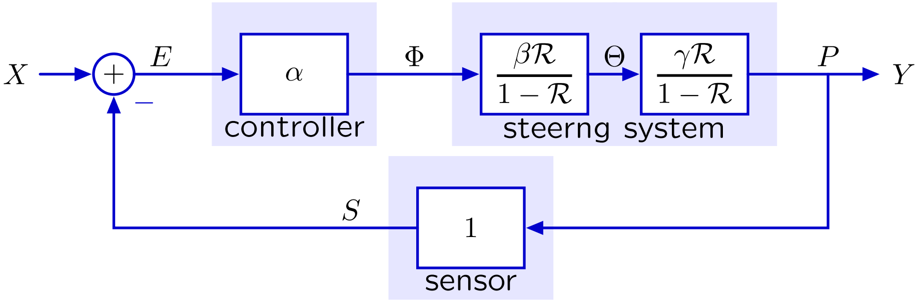
The output oscillates: poles off the real axis.

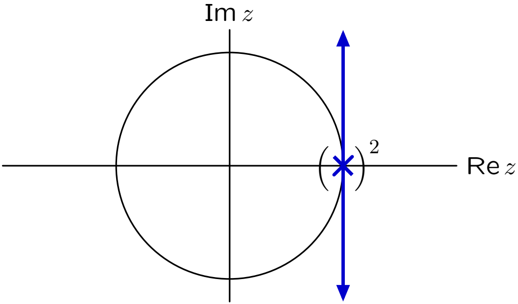
The output diverges: magnitude of poles greater than 1.

Bigger $K \rightarrow$ even less stable.

Steering Controller

No value of $K = \alpha\beta\gamma$ gives acceptable performance.

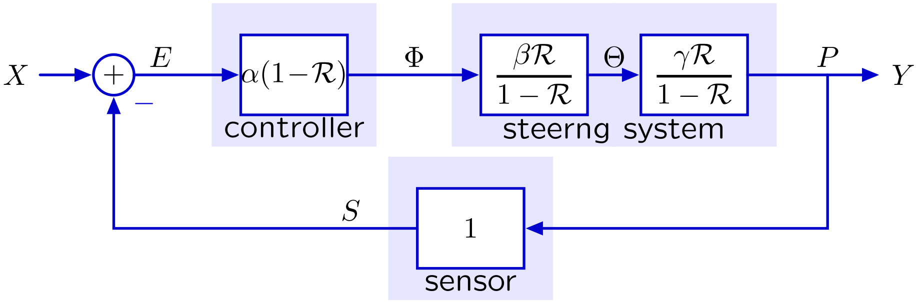




Need a better controller.

Steering Controller

Try a controller based on first differences.

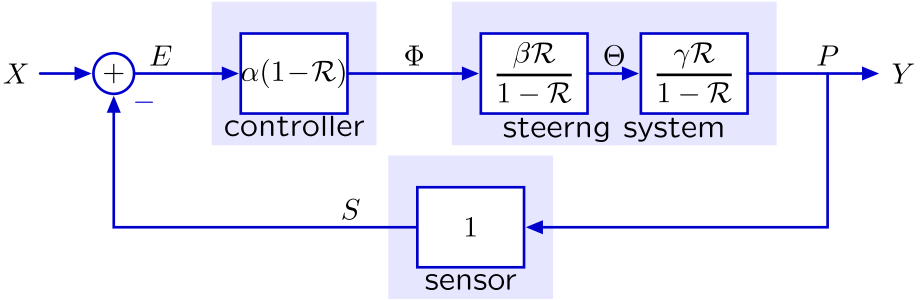


$$Y = \frac{\alpha\beta\gamma(1-\mathcal{R})\mathcal{R}^2}{(1-\mathcal{R})(1-\mathcal{R})}\,(X-Y)$$

This controller leads to a very simple form: the $(1-\mathcal{R})$ terms cancel!

Steering Controller

Try a controller based on first differences.



$$Y = \frac{\alpha\beta\gamma(1-\mathcal{R})\mathcal{R}^2}{(1-\mathcal{R})(1-\mathcal{R})}\,(X-Y) = \frac{K\mathcal{R}^2}{1-\mathcal{R}}\,(X-Y) \qquad \text{where} \quad K=\alpha\beta\gamma.$$

Solving,

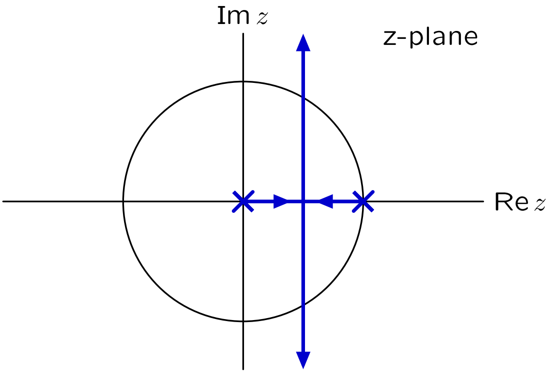
$$\frac{Y}{X} = \frac{K\mathcal{R}^2}{1-\mathcal{R}+K\mathcal{R}^2}$$

Steering Controller

The functional has the form of the "simple car" (last lecture).

$$\frac{Y}{X} = \frac{K\mathcal{R}^2}{1 - \mathcal{R} + K\mathcal{R}^2}$$

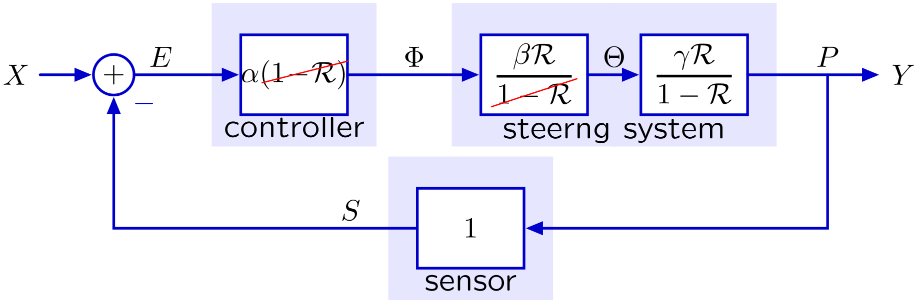
The poles are at $z = \dfrac{1 \pm \sqrt{1 - 4K}}{2}$.



The system are stable for $0 < K < 1$.

Steering Controller

Can we really cancel the $(1-\mathcal{R})$ terms in numerator and denominator?



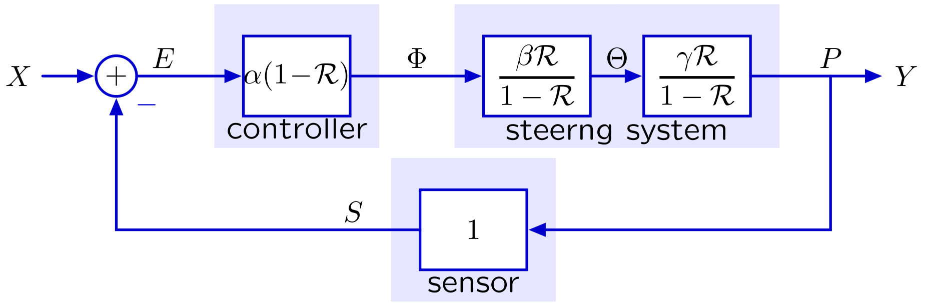
$$Y = \frac{\alpha\beta\gamma(1-\mathcal{R})\mathcal{R}^2}{(1-\mathcal{R})(1-\mathcal{R})}(X-Y) = \frac{\alpha\beta\gamma\mathcal{R}^2}{1-\mathcal{R}}(X-Y)$$

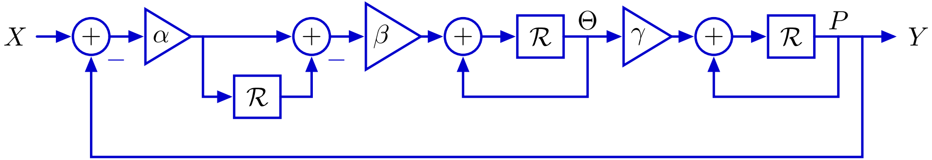
How could you test whether this is a valid operation?

Steering Controller

Simulate the behavior of the system.

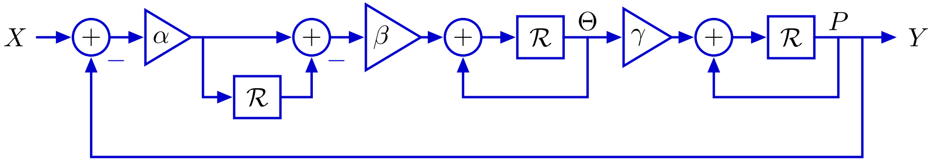
Start by making a model with just adders, gains, and delays.

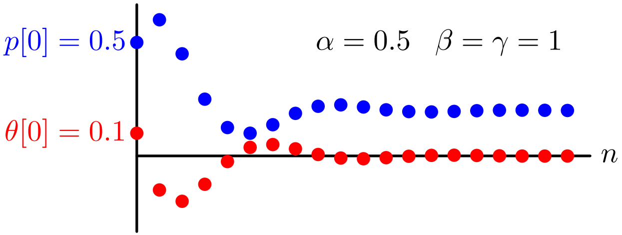




Steering Controller

Use step-by-step analysis.

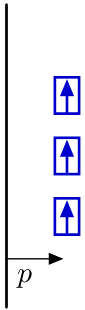




With time, the value of θ goes to zero, but the value of p does not.

Steering Controller

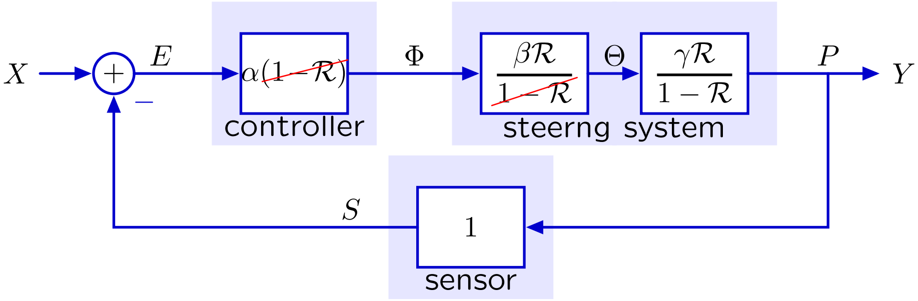
The fact that the position does not decay to zero suggests that the closed-loop response has a pole at $z = 1$.



Pole at $z = 1 \rightarrow$ fundamental mode of 1^n, $n \geq 0$.

Steering Controller

But this is not consistent with our analysis after cancelling terms.



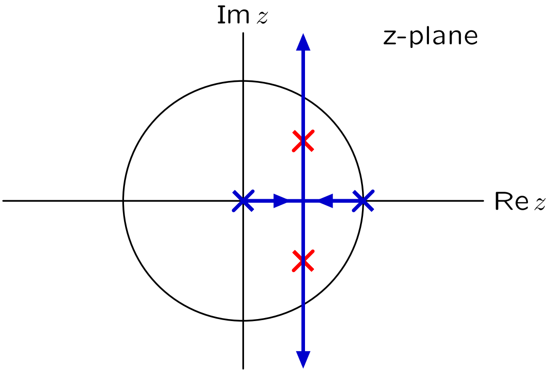
$$Y = \alpha(1-\mathcal{R})\frac{\beta\mathcal{R}}{1-\mathcal{R}}\frac{\gamma\mathcal{R}}{1-\mathcal{R}}E = \frac{\alpha\beta\gamma\mathcal{R}^2}{1-\mathcal{R}}(X-Y)$$

Solving:

$$\frac{Y}{X} = \frac{K\mathcal{R}^2}{1-\mathcal{R}+K\mathcal{R}^2} \qquad \text{where} \quad K = \alpha\beta\gamma$$

Steering Controller

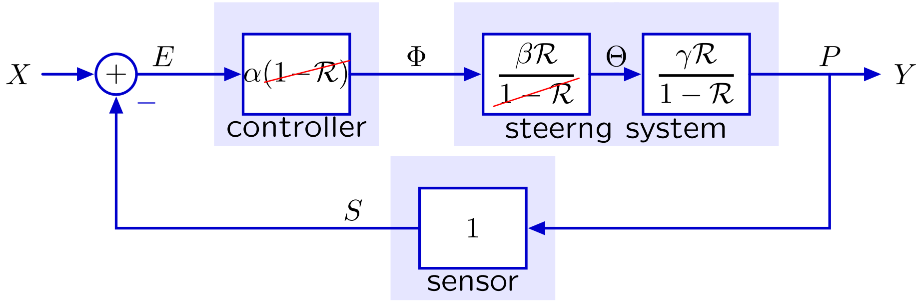
If $K = \frac{1}{2}$, the poles are at $z = \frac{1 \pm \sqrt{1-4K}}{2} = \frac{1}{2} \pm j\frac{1}{2}$.

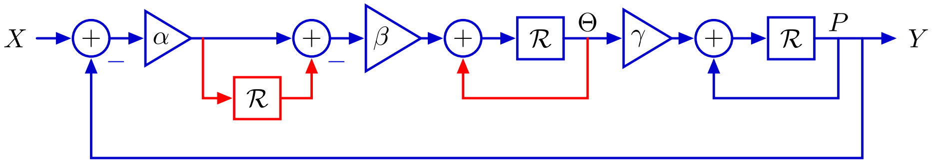


These poles correspond to decaying fundamental modes.

Steering Controller

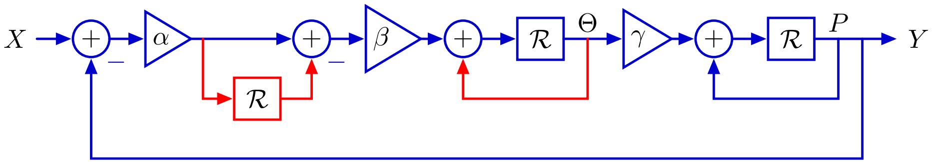
Now remove the parts that correspond to cancelled terms (red).

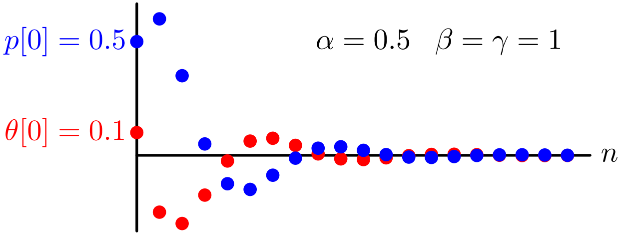




Steering Controller

Simulate again, after removing cancelled parts (simulate blue part).





Now the value of p also goes to zero with time.

Cancelling Factors in the Numerator and Denominator

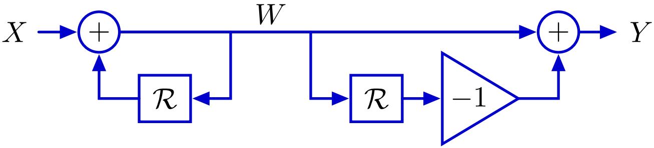
Analysis of the block diagrams before and after cancelling factors in the numerator and denominator gave different results.

Why?

Consider some simpler systems.

Cancelling Factors in the Numerator and Denominator

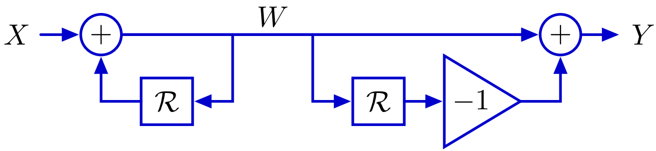
Accumulate then take difference.



$$\frac{Y}{X} = \frac{1 - \mathcal{R}}{1 - \mathcal{R}} = 1 \ ?$$

Cancelling Factors in the Numerator and Denominator

Accumulate then take difference.



$$\frac{Y}{X} = \frac{1-\mathcal{R}}{1-\mathcal{R}} = 1 \ ?$$

Think through the steps carefully:

$$W = X + \mathcal{R}W$$

$$(1-\mathcal{R})W = X$$

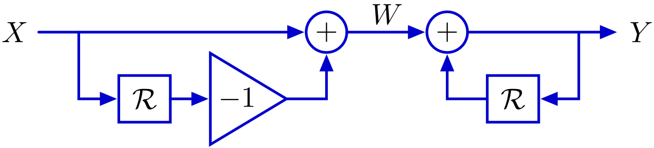
$$Y = (1-\mathcal{R})W$$

$$Y = X$$

Outputs before and after cancelling $(1-\mathcal{R})$ are the same.

Cancelling Factors in the Numerator and Denominator

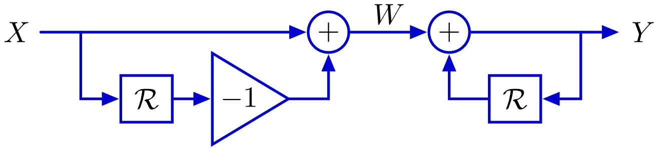
Take difference then accumulate.



$$\frac{Y}{X} = \frac{1-\mathcal{R}}{1-\mathcal{R}} = 1 \ ?$$

Cancelling Factors in the Numerator and Denominator

Take difference then accumulate.



$$\frac{Y}{X} = \frac{1-\mathcal{R}}{1-\mathcal{R}} = 1 \ ?$$

Again, think through the steps:

$$W = (1-\mathcal{R})X$$

$$Y = W + \mathcal{R}Y$$

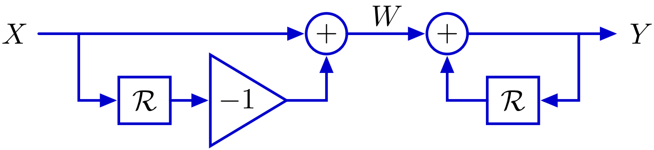
$$(1-\mathcal{R})Y = W$$

$$(1-\mathcal{R})Y = (1-\mathcal{R})X$$

Not quite the same as $Y = X$.

Cancelling Factors in the Numerator and Denominator

If system is initially at rest, then $Y = X$.



$$(1-\mathcal{R})X = (1-\mathcal{R})Y$$

$$y[0] = x[0] = 0$$

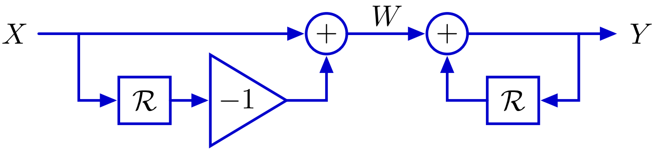
$$y[1] - y[0] = x[1] - x[0] \rightarrow y[1] = x[1]$$

$$y[2] - y[1] = x[2] - x[1] \rightarrow y[2] = x[2]$$

$$\ldots$$

Cancelling Factors in the Numerator and Denominator

If system is not initially at rest, then Y and X are not always equal.



$$(1-\mathcal{R})X = (1-\mathcal{R})Y$$

$$y[0] = 5 \qquad x[0] = 0$$

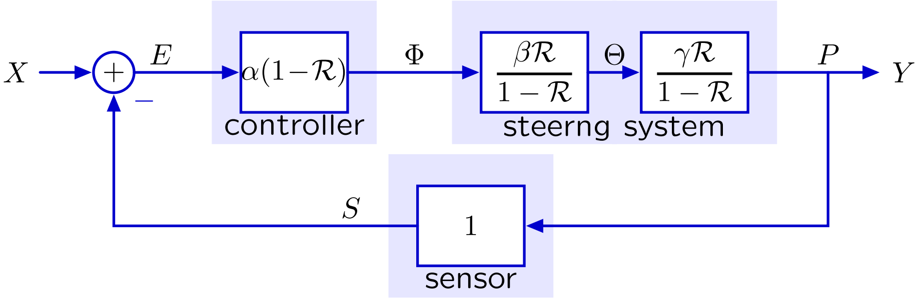
$$y[1] - y[0] = x[1] - x[0] \rightarrow y[1] = x[1] + (y[0] - x[0]) = x[1] + 5$$

$$y[2] - y[1] = x[2] - x[1] \rightarrow y[2] = x[2] + (y[1] - x[1]) = x[2] + 5$$

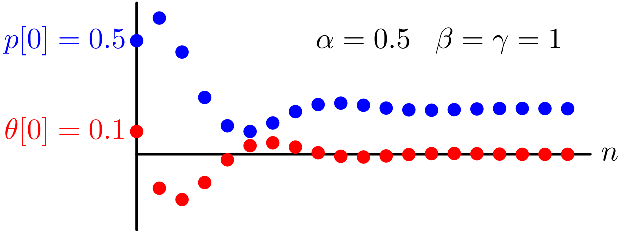
$$\ldots$$

Steering Controller

Our steering controller was not turned on at rest.

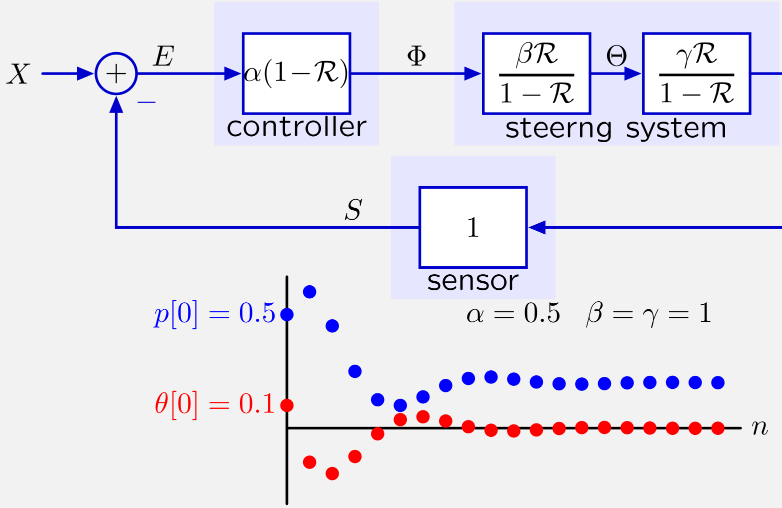


There were non-zero initial values of P and Θ.



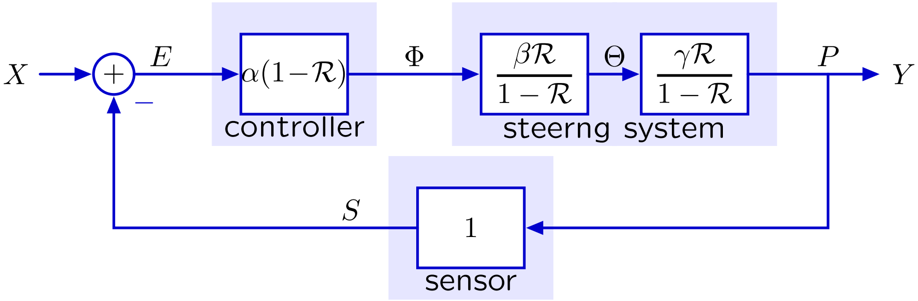
Check Yourself

Why didn't feedback make $p \to 0$?



Check Yourself

Why didn't feedback make $p \to 0$?

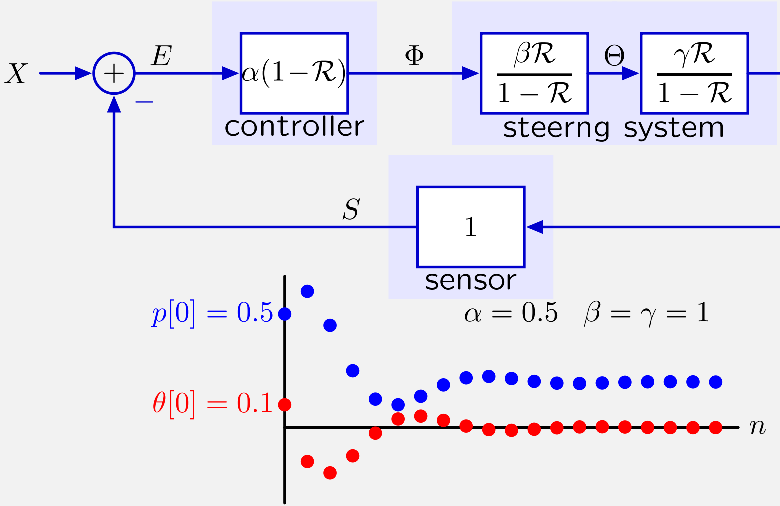


$x[n] = 0 \;\; \to \;\; e[n] = -p[n] \;\; \to \;\; \phi[n] \propto (p[n] - p[n-1])$

If $p \to$ constant, then $\phi \to 0$.

Check Yourself

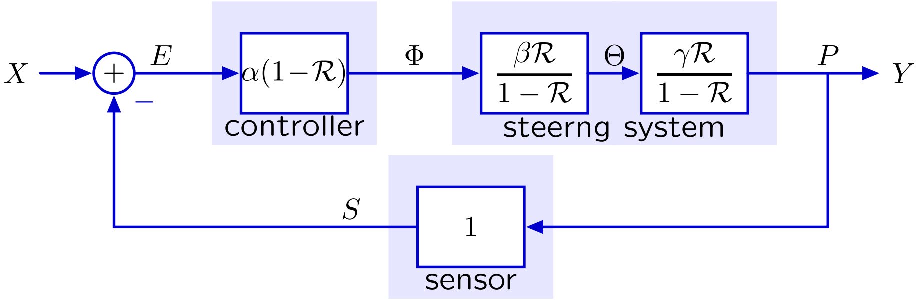
Will feedback make $\theta \to 0$?



$\alpha = 0.5 \quad \beta = \gamma = 1$

Check Yourself

Will feedback make $\theta \to 0$?



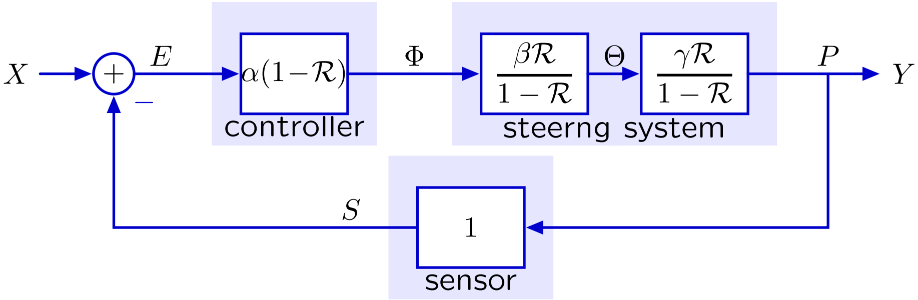
$\phi = 0$ does not imply that $\theta = 0$ (e.g., nonzero initial condition).

$\theta \neq 0 \;\to\; p$ increases $\;\to\; \phi \neq 0$.

Steering Controller

The "difference" controller is insensitive to constant inputs.

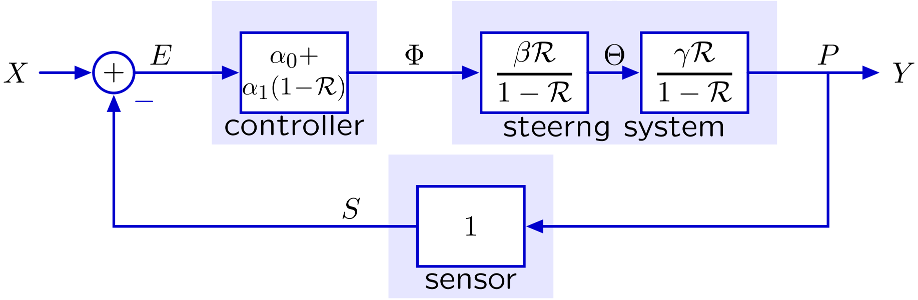
Such controllers are not good for feedback systems that are intended to control position.



Need an even better controller.

Steering Controller

Use combination of proportional and difference control to eliminate unwanted persistent behavior.



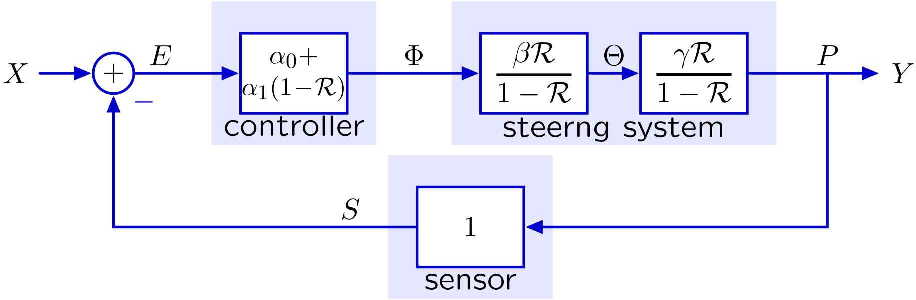
$$Y = \frac{(\alpha_0 + \alpha_1 - \alpha_1\mathcal{R})\beta\gamma\mathcal{R}^2}{(1-\mathcal{R})(1-\mathcal{R})}(X - Y)$$

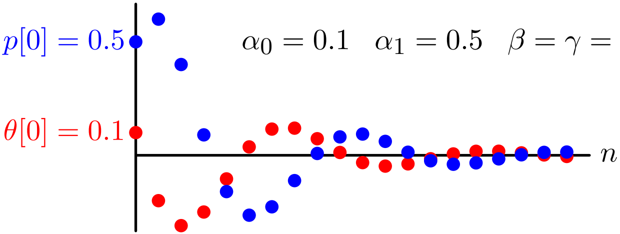
Solving:

$$\frac{Y}{X} = \frac{(\alpha_0 + \alpha_1 - \alpha_1\mathcal{R})\beta\gamma\mathcal{R}^2}{1 - 2\mathcal{R} + (1 + (\alpha_0 + \alpha_1)\beta\gamma)\mathcal{R}^2 - \alpha_1\beta\gamma\mathcal{R}^3}$$

Steering Controller

Try step-by-step analysis.





Feedback and Control: Summary

Now you know about two kinds of controllers:

- proportional
- proportional plus difference

Adding delays in a loop tends to destabilize the loop.

Adding accumulators in a loop tends to destabilize the loop.

Difference feedback can help to stabilize such loops.

Analyzing the poles of a system is key to controller design !