Frequency Response
Mid-term Examination #2

Wednesday, October 28, 7:30-9:30pm, Walker Memorial.

No recitations on the day of the exam.

Coverage: cumulative with more emphasis on recent material
lectures 1–12
homeworks 1–7

Homework 7 will include practice problems for mid-term 2. However, it will not collected or graded. Solutions will be posted.

Closed book: 2 page of notes (8½ \times 11 inches; front and back).

Designed as 1-hour exam; two hours to complete.

Review sessions during open office hours.

Conflict? Contact freeman@mit.edu by Friday, October 23, 5pm.
Review

Last time, we saw how a linear, time-invariant (LTI) system can be characterized by its unit-sample/impulse response.

\[
\text{DT: } y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]
\]

\[
\text{CT: } y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau
\]

Characterizing a system by its unit-sample/impulse response is especially insightful for some systems.
Microscope

Blurring can be represented by convolving the image with the optical “point-spread-function” (3D impulse response).

\[
\text{target image} \ast \text{image} = \text{target image}
\]

Blurring is inversely related to the diameter of the lens.
Hubble Space Telescope

Hubble images before and after upgrading the optics.

before

after

http://hubblesite.org
Frequency Response

Today we will investigate a different way to characterize a system: the **frequency response**.

Many systems are naturally described by their responses to sinusoids.

Example: audio systems
Check Yourself

How were frequencies modified in following music clips?

- **HF**: high frequencies
  - ↑: increased
- **LF**: low frequencies
  - ↓: decreased

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### Question

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If the input to a linear, time-invariant system is an eternal sinusoid, then the output is also an eternal sinusoid:

- same frequency
- possibly different amplitude, and
- possibly different phase angle.

\[ x(t) = \cos(\omega t) \]

\[ y(t) = M \cos(\omega t + \phi) \]

The **frequency response** is a plot of the magnitude \( M \) and angle \( \phi \) as a function of frequency \( \omega \).
Example

Mass, spring, and dashpot system.
Demonstration

Measure the frequency response of a mass, spring, dashpot system.
Frequency Response

Calculate the frequency response.

Methods

- solve differential equation
  → find particular solution for $x(t) = \cos \omega_0 t$
- find impulse response of system
  → convolve with $x(t) = \cos \omega_0 t$

New method

- use eigenfunctions and eigenvalues
Eigenfunctions

If the output signal is a scalar multiple of the input signal, we refer to the signal as an eigenfunction and the scale multiplier as the eigenvalue.
Consider the system described by

\[
\dot{y}(t) + 2y(t) = x(t).
\]

Determine if each of the following functions is an eigenfunction of this system. If it is, find its eigenvalue.

1. \(e^{-t}\) for all time
2. \(e^{t}\) for all time
3. \(e^{jt}\) for all time
4. \(\cos(t)\) for all time
5. \(u(t)\) for all time
Check Yourself: Eigenfunctions

\[ \dot{y}(t) + 2y(t) = x(t) \]

1. \( e^{-t} \): \[ -\lambda e^{-t} + 2\lambda e^{-t} = e^{-t} \rightarrow \lambda = 1 \]

2. \( e^{t} \): \[ \lambda e^{t} + 2\lambda e^{t} = e^{t} \rightarrow \lambda = \frac{1}{3} \]

3. \( e^{jt} \): \[ j\lambda e^{jt} + 2\lambda e^{jt} = e^{jt} \rightarrow \lambda = \frac{1}{j + 2} \]

4. \( \cos t \): \[ -\lambda \sin t + 2\lambda \cos t = \cos t \rightarrow \text{not possible!} \]

5. \( u(t) \): \[ \lambda \delta(t) + 2\lambda u(t) = u(t) \rightarrow \text{not possible!} \]
Consider the system described by
\[ \dot{y}(t) + 2y(t) = x(t). \]

Determine if each of the following functions is an eigenfunction of this system. If it is, find its eigenvalue.

1. \( e^{-t} \) for all time \( \sqrt{\lambda = 1} \)
2. \( e^{t} \) for all time \( \sqrt{\lambda = \frac{1}{3}} \)
3. \( e^{jt} \) for all time \( \sqrt{\lambda = \frac{1}{j+2}} \)
4. \( \cos(t) \) for all time \( \times \)
5. \( u(t) \) for all time \( \times \)
Complex Exponentials

Complex exponentials are eigenfunctions of LTI systems.

If $x(t) = e^{st}$ and $h(t)$ is the impulse response then

$$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = H(s)e^{st}$$

Furthermore, the eigenvalue is $H(s)$!
Rational System Functions

If a system is represented by a linear differential equation with constant coefficients, then its system function is a ratio of polynomials in $s$.

Example:

$$\ddot{y}(t) + 3\dot{y}(t) + 4y(t) = 2\ddot{x}(t) + 7\dot{x}(t) + 8x(t)$$

Then

$$H(s) = \frac{2s^2 + 7s + 8}{s^2 + 3s + 4} \equiv \frac{N(s)}{D(s)}$$
Vector Diagrams

The value of $H(s)$ at a point $s = s_0$ can be determined graphically using vectorial analysis.

Factor the numerator and denominator of the system function to make poles and zeros explicit.

$H(s_0) = K \frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2) \cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2) \cdots}$

Each factor in the numerator/denominator corresponds to a vector from a zero/pole (here $z_0$) to $s_0$, the point of interest in the $s$-plane.
Example: Find the response of the system described by

\[ H(s) = \frac{1}{s + 2} \]

to the input \( x(t) = e^{2jt} \) (for all time).

The denominator of \( H(s) \) at \( s = 2j \) is \( 2j + 2 \), a vector with length \( 2\sqrt{2} \) and angle \( \pi/4 \). Therefore, the response of the system is

\[ y(t) = H(2j)e^{2jt} = \frac{1}{2\sqrt{2}}e^{-j\pi/4}e^{2jt} . \]
Vector Diagrams

The value of $H(s)$ at a point $s = s_0$ can be determined by combining the contributions of the vectors associated with each of the poles and zeros.

$$H(s_0) = K \frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2) \cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2) \cdots}$$

The magnitude is determined by the product of the magnitudes.

$$|H(s_0)| = |K| \frac{|(s_0 - z_0)|| (s_0 - z_1)|| (s_0 - z_2)| \cdots}{|(s_0 - p_0)|| (s_0 - p_1)|| (s_0 - p_2)| \cdots}$$

The angle is determined by the sum of the angles.

$$\angle H(s_0) = \angle K + \angle (s_0 - z_0) + \angle (s_0 - z_1) + \cdots - \angle (s_0 - p_0) - \angle (s_0 - p_1) - \cdots$$
Response to eternal sinusoids.

Let \( x(t) = \cos \omega_0 t \) (for all time). Then

\[
x(t) = \frac{1}{2} \left( e^{j\omega_0 t} + e^{-j\omega_0 t} \right)
\]

and the response to a sum is the sum of the responses.

\[
y(t) = \frac{1}{2} \left( H(j\omega_0) e^{j\omega_0 t} + H(-j\omega_0) e^{-j\omega_0 t} \right)
\]
Conjugate Symmetry

The complex conjugate of $H(j\omega)$ is $H(-j\omega)$.

The system function is the Laplace transform of the impulse response:

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st} dt$$

where $h(t)$ is a real-valued function of $t$ for physical systems.

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt$$

$$H(-j\omega) = \int_{-\infty}^{\infty} h(t)e^{j\omega t} dt \equiv (H(j\omega))^*$$
Frequency Response

Response to eternal sinusoids.

Let $x(t) = \cos \omega_0 t$ (for all time), which can be written as

$$x(t) = \frac{1}{2} \left( e^{j\omega_0 t} + e^{-j\omega_0 t} \right)$$

The response to a sum is the sum of the responses,

$$y(t) = \frac{1}{2} \left( H(j\omega_0)e^{j\omega_0 t} + H(-j\omega_0)e^{-j\omega_0 t} \right)$$

$$= \text{Re} \left\{ H(j\omega_0)e^{j\omega_0 t} \right\}$$

$$= \text{Re} \left\{ |H(j\omega_0)|e^{j\angle H(j\omega_0)}e^{j\omega_0 t} \right\}$$

$$= |H(j\omega_0)| \text{Re} \left\{ e^{j\omega_0 t + j\angle H(j\omega_0)} \right\}$$

$$y(t) = |H(j\omega_0)| \cos (\omega_0 t + \angle (H(j\omega_0))).$$
Frequency Response

The magnitude and phase of the response of a system to an eternal cosine signal is the magnitude and phase of the system function evaluated at \( s = j\omega \).

\[
\cos(\omega t) \rightarrow H(s) \rightarrow |H(j\omega)| \cos(\omega t + \angle H(j\omega))
\]
Vector Diagrams

\[ H(s) = s - z_1 \]

- Plane

\[ |H(j\omega)| \]

\[ \angle H(j\omega) \]
Vector Diagrams

\[ H(s) = s - z_1 \]
Vector Diagrams

\[ H(s) = s - z_1 \]
Vector Diagrams

\[ H(s) = s - z_1 \]

\[ |H(j\omega)| \]

\[ \angle H(j\omega) \]

$s$-plane

\[ s \]

\[ \sigma \]

\[ \omega \]
Vector Diagrams

\[ H(s) = s - z_1 \]

- **s-plane**
  - \( \omega \)
  - \( s \)

- **Magnitude** \( |H(j\omega)| \)
  - \( \pi/2 \)
  - \( -\pi/2 \)

- **Angle** \( \angle H(j\omega) \)
  - \( 0 \)
  - \( 5 \)
Vector Diagrams

\[ H(s) = s - z_1 \]

\[ s\text{-plane} \]

\[ |H(j\omega)| \]

\[ \angle H(j\omega) \]
Vector Diagrams

\[ H(s) = s - z_1 \]
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\[ H(s) = s - z_1 \]
Vector Diagrams

\[ H(s) = s - z_1 \]
\[ H(s) = \frac{9}{s - p_1} \]

Diagram showing the \( s \)-plane with the transfer function \( H(s) \) and its magnitude and phase responses.

\[ |H(j\omega)| \]

\[ \angle H(j\omega) \]

\( \sigma \) and \( \omega \) axes are labeled as \( s \)-plane.
Vector Diagrams

\[ H(s) = \frac{9}{s - p_1} \]
Vector Diagrams

\[ H(s) = \frac{9}{s - p_1} \]

\( s \)-plane

\[ |H(j\omega)| \]

\( \angle H(j\omega) \)

\(-5\) \(\sigma\) \(5\)

\(-5\) \(\omega\) \(5\)

\(-5\) \(|H(j\omega)|\) \(5\)

\(-\pi/2\) \((-\pi/2)\)

\(\pi/2\)
Vector Diagrams

\[ H(s) = \frac{9}{s - p_1} \]
Vector Diagrams

\[ H(s) = \frac{9}{s - p_1} \]

\( s \)-plane

\[ |H(j\omega)| \]

\[ \angle H(j\omega) \]
Vector Diagrams

\[ H(s) = \frac{9}{s - p_1} \]
Vector Diagrams

\[ H(s) = 3 \frac{s - z_1}{s - p_1} \]
Vector Diagrams

\[ H(s) = 3 \frac{s - z_1}{s - p_1} \]

\[ \angle H(j\omega) \]

| \( H(j\omega) \) |
Vector Diagrams

\[ H(s) = 3 \frac{s - z_1}{s - p_1} \]

$s$-plane

\[ |H(j\omega)| \]

\[ \angle H(j\omega) \]
Vector Diagrams

\[ H(s) = 3 \frac{s - z_1}{s - p_1} \]
\[ H(s) = 3 \frac{s - z_1}{s - p_1} \]
Vector Diagrams

\[ H(s) = 3 \frac{s - z_1}{s - p_1} \]

\[ |H(j\omega)| \]

\[ \angle H(j\omega) \]
Example: Mass, Spring, and Dashpot

\[ F = Ma = M\ddot{y}(t) = K(x(t) - y(t)) - B\dot{y}(t) \]

\[ M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = Kx(t) \]

\[ (s^2M + sB + K) \ Y(s) = KX(s) \]

\[ H(s) = \frac{K}{s^2M + sB + K} \]
Vector Diagrams

\[ H(s) = \frac{15}{(s - p_1)(s - p_2)} \]

$s$-plane

\[ |H(j\omega)| \]

\[ \angle H(j\omega) \]
Vector Diagrams

\[ H(s) = \frac{15}{(s - p_1)(s - p_2)} \]

\[ |H(j\omega)| \]

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Vector Diagrams

\[ H(s) = \frac{15}{(s - p_1)(s - p_2)} \]
Vector Diagrams

\[ H(s) = \frac{15}{(s - p_1)(s - p_2)} \]

\( s \)-plane

\[ |H(j\omega)| \]

\[ \angle H(j\omega) \]
Vector Diagrams

\[ H(s) = \frac{15}{(s - p_1)(s - p_2)} \]

\[ s\text{-plane} \]

\[ |H(j\omega)| \]

\[ \angle H(j\omega) \]
The transfer function $H(s)$ is given by:

$$H(s) = \frac{15}{(s - p_1)(s - p_2)}$$

The magnitude $|H(j\omega)|$ and angle $\angle H(j\omega)$ are plotted in the $s$-plane.
Check Yourself

Consider the system represented by the following poles.

\[ s\text{-plane} \]

\[ -\sigma \]

\[ -\omega_d \]

\[ \omega_d \]

\[ \omega_0 \]

Find the frequency \( \omega \) at which the magnitude of the response \( y(t) \) is greatest if \( x(t) = \cos \omega t \).

1. \( \omega = \omega_d \)
2. \( \omega_d < \omega < \omega_0 \)
3. \( 0 < \omega < \omega_d \)
4. none of the above
Analyze with vectors.

The product of the lengths is \( \left( \sqrt{(\omega + \omega_d)^2 + \sigma^2} \right) \left( \sqrt{(\omega - \omega_d)^2 + \sigma^2} \right) \).
Check Yourself: Frequency Response

Analyze with vectors.

The product of the lengths is \( \left( \sqrt{(\omega + \omega_d)^2 + \sigma^2} \right) \left( \sqrt{(\omega - \omega_d)^2 + \sigma^2} \right) \).

Decreasing \( \omega \) from \( \omega_d \) to \( \omega_d - \epsilon \) decreases the product since length of bottom vector decreases as \( \epsilon \) while length of top vector increases only \( \epsilon^2 \).
More mathematically ...

The product of the lengths is \( \left( \sqrt{(\omega + \omega_d)^2 + \sigma^2} \right) \left( \sqrt{(\omega - \omega_d)^2 + \sigma^2} \right) \).

Maximum occurs where derivative of squared lengths is zero.

\[
\frac{d}{d\omega} \left( (\omega + \omega_d)^2 + \sigma^2 \right) \left( (\omega - \omega_d)^2 + \sigma^2 \right) = 0
\]

\[
\rightarrow \quad \omega^2 = \omega_d^2 - \sigma^2 = \omega_0^2 - 2\sigma^2.
\]
Consider the system represented by the following poles. 

\[ s\text{-plane} \]

\[ \omega_0, -\sigma, -\omega_d, \omega \]

Find the frequency \( \omega \) at which the magnitude of the response \( y(t) \) is greatest if \( x(t) = \cos \omega t \).

1. \( \omega = \omega_d \)
2. \( \omega_d < \omega < \omega_0 \)
3. \( 0 < \omega < \omega_d \)
4. none of the above
Consider the system represented by the following poles.

Find the frequency $\omega$ at which the phase of the response $y(t)$ is $-\pi/2$ if $x(t) = \cos \omega t$.

0. $0 < \omega < \omega_d$
1. $\omega = \omega_d$
2. $\omega_d < \omega < \omega_0$
3. $\omega = \omega_0$
4. $\omega > \omega_0$
5. none
Check Yourself

The phase is 0 when $\omega = 0$. 

\[ s\text{-plane} \]

\[ \omega_0, \omega_d, -\sigma, -\omega_d \]
The phase is less than $\pi/2$ when $\omega = \omega_d$. 

![Diagram showing the phase relation in the s-plane with $\omega_0$, $\omega_d$, $-\sigma$, $-\omega_d$, and $\alpha$.](image)
The phase is less than $\pi/2$ when $\omega = \omega_d$. 

![Diagram showing the phase relationship between $\omega_0$ and $\omega_d$ in the s-plane.]

**Check Yourself**
Check Yourself

The phase at $\omega = \omega_0$ is $-\pi/2$. 

\[ \omega_0 - \sigma \omega_d - \omega_d \alpha \beta \omega_0 - \sigma \omega_d - \omega_d \pi/2 \]
Consider the system represented by the following poles.

Find the frequency $\omega$ at which the phase of the response $y(t)$ is $-\pi/2$ if $x(t) = \cos \omega t$.

- 0. $0 < \omega < \omega_d$
- 1. $\omega = \omega_d$
- 2. $\omega_d < \omega < \omega_0$
- 3. $\omega = \omega_0$
- 4. $\omega > \omega_0$
- 5. none
Frequency Response: Summary

LTI systems can be characterized by responses to eternal sinusoids.

Many systems are naturally described by their frequency response.
– audio systems
– mass, spring, dashpot system

Frequency response is easy to calculate from the system function.