RADIATION BY CHARGE ρ AND CURRENT $\bar{J}$

We know EM waves exist

$$\text{Maxwell's Equation} \Rightarrow \sum_{\text{Direction, Pol.}}^{\infty} \text{U.P.W}$$

But how do we create waves, and "radiate"?

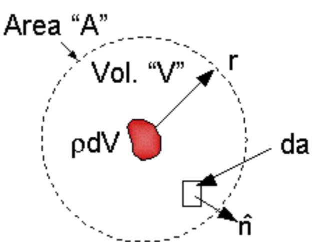
Maxwell's Equations suggest we can change $\bar{E}$, $\bar{H}$ by moving charges

Our approach to deriving radiation equations:

- Assume $\rho, \bar{J}$; solve for $\bar{E}, \bar{H}$
- First statics, then dynamics

L4-1

ELECTROSTATIC EQUATIONS



$$\nabla \times \bar{E} = \frac{-\partial \bar{B}}{\partial t} = 0 \qquad \nabla \bullet \bar{E} = \frac{\rho}{\varepsilon_o}$$

Faraday's Law Gauss's Law

$$\text{Integral form of Gauss's Law} \rightarrow \int_A \bar{E} \bullet \hat{n}\, da = \int_V \frac{\rho}{\varepsilon_o} dv$$

$$\left(\text{Recall: } \nabla \equiv \hat{x}\partial/\partial x + \hat{y}\partial/\partial y + \hat{z}\partial/\partial z\right)$$

Let $V \rightarrow 0$, be spherical $\Rightarrow \bar{E}(\bar{r}) = \hat{r}E(r)$ since problem and solution are spherically symmetric

From Gauss's Law: $4\pi r^2 \bar{E}(r) \bullet \hat{r} = \rho dV/\varepsilon_o$ for a sphere of radius r.

$$\text{Solving} \Rightarrow \boxed{\bar{E}(r) = \hat{r}\frac{\rho dV}{4\pi\varepsilon_o r^2}}$$

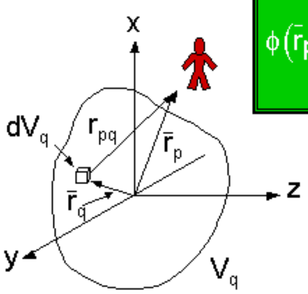
L4-2

ELECTROSTATIC EQUATIONS

Since $\nabla \times \bar{E} = 0$ $\boxed{\bar{E} = -\nabla\phi\,,\ \phi \text{ is scalar potential}}$ $[\nabla \times (-\nabla\phi) \equiv 0]$

Therefore $\boxed{\phi(r) = \int_r^{\infty} \bar{E} \bullet \hat{r}dr = \dfrac{\rho dV}{4\pi\varepsilon_o r}}$ we define $\phi = 0$ as $r \to \infty$ $\left(\text{Here } \nabla\phi = \hat{r}\dfrac{\partial\phi}{\partial r}\right)$

Solution by superposition



$$\boxed{\phi(\bar{r}_p) = \int_{V_q} \frac{\rho(\bar{r}_q)}{4\pi\varepsilon_o r_{pq}} dV_q}$$ solution to Poisson's Equation

NOTE:
$$\boxed{\begin{aligned} \nabla\bullet\bar{E} &= \nabla\bullet(-\nabla\phi) = \\ -\nabla^2\phi &= \rho/\varepsilon \quad \text{is Poisson's Equation} \\ \nabla^2\phi &= 0 \quad \text{is Laplace's Equation} \end{aligned}}$$

L4-3

MAGNETOSTATIC EQUATIONS

Maxwell's Equations:

When $\dfrac{\partial}{\partial t} = 0,\ \nabla \times \bar{H} = \bar{J}\,,\ \nabla\bullet\bar{B} = 0$

Let $\bar{B} \triangleq \nabla \times \bar{A}$

This satisfies $\nabla\bullet\bar{B} = 0$ $\left[\nabla\bullet(\nabla \times \bar{A}) \equiv 0\,,\ \text{all } \bar{A}\right]$

NOTE: This does NOT fully define $\bar{A}$

Claim: $\nabla\bullet\bar{A}$, $\nabla \times \bar{A}$ can be fixed independently

Ampere's Law becomes:

$$\nabla \times \bar{B} = \nabla \times (\nabla \times \bar{A}) = \mu_o \bar{J}$$
$$= \nabla \underbrace{(\nabla\bullet\bar{A})}_{[\nabla\bullet\bar{A} \triangleq 0]} - \nabla^2\bar{A} = \mu_o \bar{J}$$

$$\boxed{\therefore\ \nabla^2\bar{A} = -\mu_o \bar{J} \quad \text{Vector Poisson Equation}}$$

L4-4

MAGNETOSTATIC EQUATIONS (2)

Solutions to Vector and Scalar Poisson's Equations

For $\bar{A}$: $\nabla^2(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z) = -\mu_o[\hat{x}J_x + \hat{y}J_y + \hat{z}J_z]$

For ϕ: $\nabla^2\phi = -\text{constant} \quad \Rightarrow \quad \phi_p = \int_{vol} \frac{\text{constant}}{4\pi r_{pq}} dV_q$

Therefore $\nabla^2 A_x = -\mu_o J_x \quad \Rightarrow \quad A_{x_p} = \int_{V_q} \frac{\mu_o J_{x_q}}{4\pi r_{pq}} dV_q$

So $\boxed{\bar{A}_p = \int_{V_q} \frac{\mu_o \bar{J}}{4\pi r_{pq}} dV_q \quad \text{Solution to vector Poisson equation}}$

$\boxed{\text{General solution algorithm} \quad \{\rho, \bar{J}\} \Rightarrow \{\phi, \bar{A}\} \Rightarrow \{\bar{E}, \bar{H}\}}$

RADIATION BY DYNAMIC CHARGES, $\rho(t), \bar{J}(t)$

Maxwell's Equations with sources

$$\nabla \times \bar{E} = -\partial\bar{B}/\partial t \qquad \nabla \bullet \bar{E} = \rho/\varepsilon$$
$$\nabla \times \bar{H} = \bar{J} + \partial\bar{D}/\partial t \qquad \nabla \bullet \mu\bar{H} = 0$$

4 steps

(1) Let $\boxed{\bar{B} = \nabla \times \bar{A}}$ since $\nabla \bullet \bar{B} = 0 = \nabla \bullet (\nabla \times \bar{A})$

(2) Therefore $\nabla \times \bar{E} = -\frac{\partial}{\partial t}(\nabla \times \bar{A})$ and $\nabla \times \underbrace{(\bar{E} + \partial\bar{A}/\partial t)}_{-\nabla\phi} = 0$

Therefore $\boxed{\bar{E} = -\nabla\phi - \partial\bar{A}/\partial t}$

(3) $\nabla \times \bar{H} = \nabla \times (\nabla \times \bar{A})/\mu_o = \left[\nabla(\nabla \bullet \bar{A}) - \nabla^2\bar{A}\right]/\mu_o = \bar{J} + \varepsilon_o \frac{\partial}{\partial t}\left[-\nabla\phi - \frac{\partial\bar{A}}{\partial t}\right]$

Therefore $-\nabla^2\bar{A} + \nabla\underbrace{\left[\nabla \bullet \bar{A} + \mu_o\varepsilon_o \frac{\partial\phi}{\partial t}\right]}_{\triangleq 0} + \mu_o\varepsilon_o \frac{\partial^2\bar{A}}{\partial t^2} = \mu_o\bar{J}$

RADIATION BY DYNAMIC CHARGES, $\rho(t),\bar{J}(t)$ (2)

$$-\nabla^2\bar{A}+\nabla\underbrace{\left[\nabla\bullet\bar{A}+\mu_o\varepsilon_o\frac{\partial\phi}{\partial t}\right]}_{\triangleq 0}+\mu_o\varepsilon_o\frac{\partial^2\bar{A}}{\partial t^2}=\mu_o\bar{J}$$

$$\nabla\bullet\bar{A}=-\mu_o\varepsilon_o\frac{\partial\phi}{\partial t}$$ "Lorentz Gauge"

before we defined only $\nabla\times\bar{A}=\bar{B}$

$$\boxed{\left[\nabla^2-\mu_o\varepsilon_o\frac{\partial^2}{\partial t^2}\right]\bar{A}=-\mu_o\bar{J}\ \ \text{(Wave equation)}}$$

$\bar{J}\Rightarrow\bar{A}$ if we solve
"Inhomogeneous vector Helmholz equation"

L4-7

"RETARDED POTENTIAL" RADIATION SOLUTION

$$\nabla\bullet\bar{E}=-\nabla\bullet\left[\nabla\phi+\frac{\partial\bar{A}}{\partial t}\right]=-\nabla^2\phi-\frac{\partial}{\partial t}\underbrace{\left(\nabla\bullet\bar{A}\right)}_{-\mu_o\varepsilon_o\frac{\partial\phi}{\partial t}}=\rho/\varepsilon$$

$$\boxed{\left[\nabla^2-\mu_o\varepsilon_o\frac{\partial^2}{\partial t^2}\right]\phi=-\rho/\varepsilon_o}$$

$\rho\Rightarrow\phi$ If we solve
"inhomogeneous scalar
Helmholz equation"

Time-Domain Solutions: wavelike versions of static solution

$$\phi_p=\frac{1}{4\pi\varepsilon_o}\int_{V_q}\frac{\rho_q\left(t-r_{pq}/c\right)}{r_{pq}}dV_q$$

$$\bar{A}_p=\frac{\mu_o}{4\pi}\int_{V_q}\frac{\bar{J}_q\left(t-r_{pq}/c\right)}{r_{pq}}dV_q$$

"Retarded potentials" $\Rightarrow$ static solution if $c\to\infty$

L4-8

SINUSOIDAL STEADY STATE SOLUTIONS

$$\phi_p = \frac{1}{4\pi\varepsilon_o}\int_{V_q}\frac{\rho_q\left(t-r_{pq}/c\right)}{r_{pq}}dV_q \qquad \bar{A}_p = \frac{\mu_o}{4\pi}\int_{V_q}\frac{\bar{J}_q\left(t-r_{pq}/c\right)}{r_{pq}}dV_q$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\underline{\bar{A}}(\omega)=\frac{\mu_o}{4\pi}\int_{V_q}\frac{\underline{\bar{J}}(\omega)}{r_{pq}}e^{-jkr_{pq}}dV_q \qquad \underline{\Phi}(\omega)=\frac{1}{4\pi\varepsilon_o}\int_{V_q}\frac{\underline{\rho}(\omega)e^{-jkr_{pq}}}{r_{pq}}dV_q$$

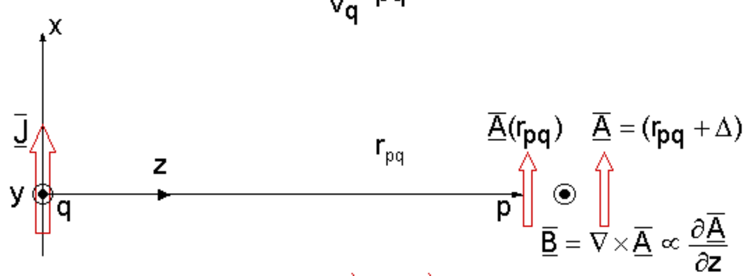
$$\underline{\bar{B}}=\nabla\times\underline{\bar{A}}\ ,\ \underline{\bar{E}}=-\nabla\underline{\Phi}-j\omega\underline{\bar{A}} \qquad k=\omega\sqrt{\mu_o\varepsilon_o}=2\pi/\lambda_o$$

Algorithm to solve general problem: $\{\rho,\bar{J}\}\Rightarrow\{\phi,\bar{A}\}\Rightarrow\{\bar{E},\bar{H}\}$

L4-9

RADIATION EXAMPLE

$$\underline{\bar{A}}(r_{pq})=\frac{\mu_o}{4\pi}\int_{V_q}\frac{\underline{\bar{J}}}{r_{pq}}e^{-jkr_{pq}}dV_q$$



Along z axis $\underline{\bar{B}}=\left(\hat{x}\cancel{\frac{\partial}{\partial x}}+\hat{y}\cancel{\frac{\partial}{\partial y}}+\hat{z}\frac{\partial}{\partial z}\right)\times\underline{\bar{A}}\neq 0$ because of Δ delay

$\cancel{}\to 0 \quad \cancel{}\to 0$

$$\underline{\bar{H}}=\underline{\bar{B}}/\mu_o$$

$$\underline{\bar{E}}=\left(\nabla\times\underline{\bar{H}}\right)/j\omega\varepsilon_o \quad \text{since} \begin{cases}\nabla\times\underline{\bar{H}}=\underline{\bar{J}}+j\omega\varepsilon_o\underline{\bar{E}}\\ \underline{\bar{J}}=0 \text{ at } p\end{cases}$$

L4-10