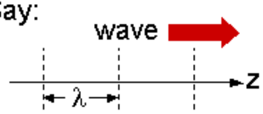
PLANE WAVES AT BOUNDARIES

Wave equation: $(\nabla^2 + \underbrace{\omega^2\mu\varepsilon}_{k^2})\overline{\underline{E}} = 0$ in source-free region

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, $k = \frac{2\pi}{\lambda} = \frac{\omega}{c} = \omega\sqrt{\mu\varepsilon}$

z-Directional Wave:

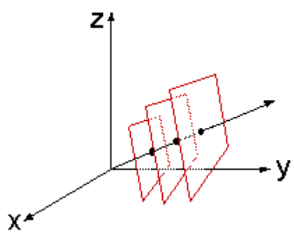
Say:



"Phase fronts"

$$\overline{\underline{E}} = \overline{\underline{E}}_o e^{-jkz}$$

3-D Wave:

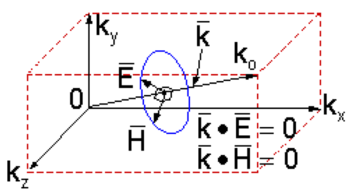


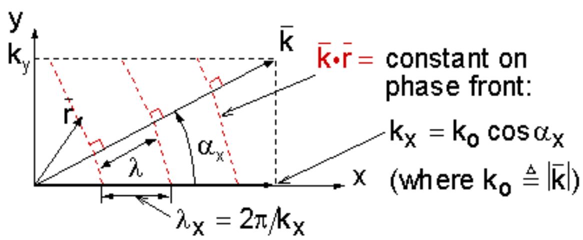
$$\overline{\underline{E}} = \overline{\underline{E}}_o e^{-jk_x x - jk_y y - jk_z z}$$

$$\overline{\underline{E}} = \overline{\underline{E}}_o e^{-j\overline{k}\cdot\overline{r}} \quad \text{where} \begin{cases} \overline{k} \triangleq \hat{x}k_x + \hat{y}k_y + \hat{z}k_z \\ \overline{r} \triangleq \hat{x}x + \hat{y}y + \hat{z}z \end{cases}$$

L7-1

WAVES PROPAGATING IN THREE DIMENSIONS





Dispersion Relation:

Substitute $\overline{\underline{E}}_o e^{-j\overline{k}\cdot\overline{r}}$ into wave equation $(\nabla^2 + k_o{}^2)\overline{\underline{E}} = 0$ where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \Rightarrow \boxed{k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\varepsilon \triangleq k_o^2} \Rightarrow$$

Wave Vector $\overline{k}$:

$$\nabla\cdot\overline{\underline{E}} = 0 = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\bullet\left(\hat{x}\underline{E}_x + \hat{y}\underline{E}_y + \hat{z}\underline{E}_z\right)e^{-j(k_x x + k_y y + k_z z)}$$

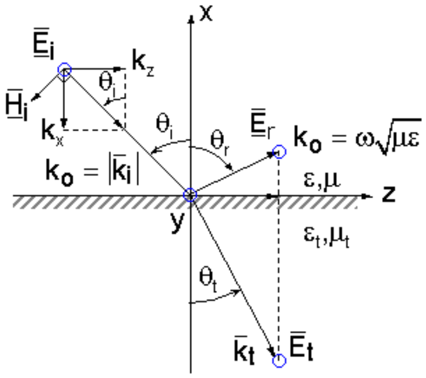
$$\Rightarrow -j(k_x\underline{E}_x + k_y\underline{E}_y + k_z\underline{E}_z) = -j\overline{k}\cdot\overline{\underline{E}} = 0, \quad \therefore \quad \overline{k} \perp \overline{\underline{E}}$$

$$\nabla\cdot\overline{\underline{H}} = -j\overline{k}\cdot\overline{\underline{H}} = 0, \quad \therefore \quad \overline{k} \perp \overline{\underline{H}}$$

L7-2

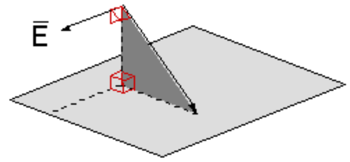
CONSIDER UPW AT PLANAR BOUNDARY

Case I: TE Wave



"Transverse Electric"

$\triangleq \bar{E} \perp$ Plane of incidence



Trial Solutions:

Incident: $\bar{E}_i = \hat{y}E_o e^{+jk_x x - jk_z z} = \hat{y}E_o e^{+jk_o \cos\theta_i x - jk_o \sin\theta_i z}$

Reflected: $\bar{E}_r = \hat{y}\underline{\Gamma}E_o e^{-jk_o \cos\theta_r x - jk_o \sin\theta_r z}$

Transmitted: $\bar{E}_t = \hat{y}\underline{T}E_o e^{jk_t \cos\theta_t x - jk_t \sin\theta_t z}$

L7-3

IMPOSE BOUNDARY CONDITIONS @ x = 0

$\bar{E}_{//}$ is continuous

At x = 0:

$$E_o e^{-\overbrace{jk_o \sin\theta_i}^{k_z} z} + \underline{\Gamma}E_o e^{-jk_o \sin\theta_r z} = \underline{T}E_o e^{-jk_t \sin\theta_t z} \quad \text{for all } z$$

Therefore $\underbrace{k_o \sin\theta_i}_{k_{i_z}} = \underbrace{k_o \sin\theta_r}_{k_{r_z}} = \underbrace{k_t \sin\theta_t}_{k_{t_z}} = k_z$

and $\theta_r = \theta_i$ Angle of incidence equals angle of reflection

Therefore: $\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_o}{k_t} = \frac{\omega\sqrt{\mu\varepsilon}}{\omega\sqrt{\mu_t\varepsilon_t}} = \frac{v_t}{v_i} = \frac{n_i}{n_t}$ "Snell's Law"

where $n \triangleq c/v_{phase} = c\sqrt{\mu\varepsilon}$ "Refractive Index"

$n_{vacuum} = 1$

$n_{glass} \cong 1.5 - 1.66$

$n_{water} \cong 1.3$ at visible wavelengths

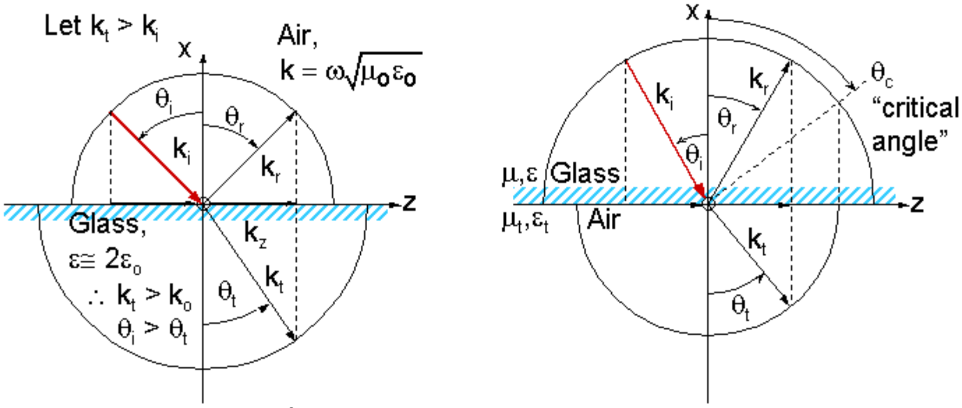
$\gtrsim 9$ at audio-radio frequencies

L7-4

ONE WAY TO VISUALIZE SNELL'S LAW

Recall: $k_o \sin\theta_i = k_t \sin\theta_t$

Let $k_t < k_i$; then $\theta_t \geq \theta_i$



We know: $\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_i}{n_t}$ But when $\sin\theta_t = 1$, then $\theta_i = \theta_c$ where

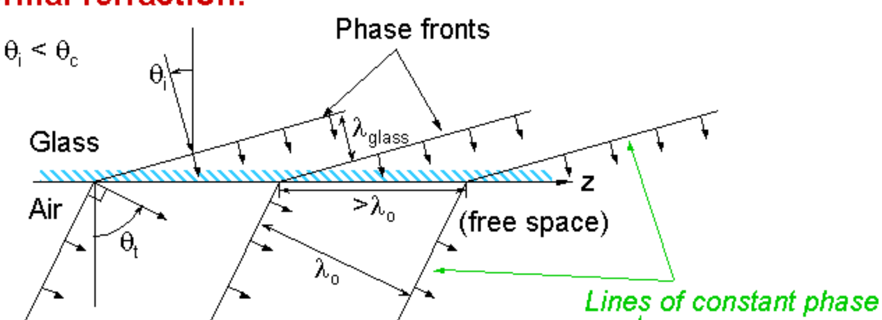
$$\theta_c = \sin^{-1}(n_t/n_i) \quad \text{"critical angle"}$$

$$\left(\text{e.g. } [\varepsilon_i = 2\varepsilon_o, \mu = \mu_o] \Rightarrow [n_i = \sqrt{2}] \Rightarrow [\theta_c = 45°]\right)$$

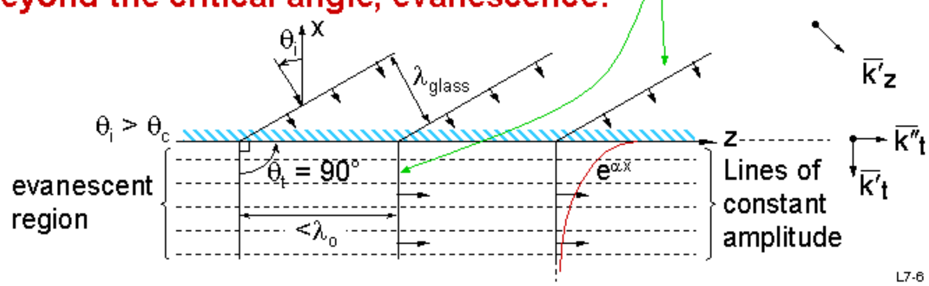
L7-5

NON-UNIFORM PLANE WAVES (NUPW) 2

Normal refraction:



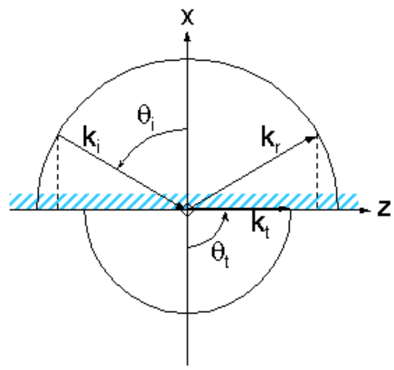
Beyond the critical angle, evanescence:



L7-6

NON-UNIFORM PLANE WAVES (2)

Beyond the critical angle:



$$\underline{\bar{E}}_t = \hat{y}\underline{E}_o e^{+\alpha x - jk_z z} \quad (x < 0)$$
$$= \hat{y}\underline{E}_o e^{-j\underline{\bar{k}}_t \cdot \bar{r}}$$

where: $\underline{\bar{k}}_t = k_z\hat{z} + j\alpha_{tx}\hat{x} \triangleq \bar{k}' - j\bar{k}''$

Called:

"non-uniform plane wave"
"evanescent wave" (no power in direction of decay)
"surface wave"
"inhomogeneous plane wave"

Therefore, for the evanescent wave:

If lossless medium $\bar{k}' \bullet \bar{k}'' = 0$ $\quad \underline{\bar{E}}, \underline{\bar{H}} \propto e^{-j(\bar{k}' - j\bar{k}'')\cdot\bar{r}}$

(general expression)

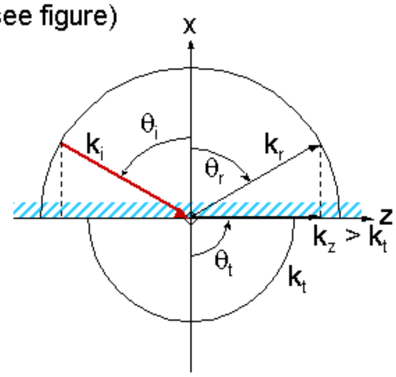
L7-7

WHAT HAPPENS WHEN θ_i > THE CRITICAL ANGLE θ_c?

Since: $k_t^2 = \omega^2\mu_t\varepsilon_t = k_z^2 + k_{tx}^2$

Therefore: $k_{tx}^2 = k_t^2 - k_z^2$, < 0 for $\theta_i > \theta_c$! (see figure)

For $\theta_i > \theta_c$ $\quad \underline{k}_{tx} = \pm j\alpha = \pm j\sqrt{k_z^2 - k_t^2}$



where: $k_z^2 = \omega^2\mu_i\varepsilon_i \sin^2\theta_i$

$k_t^2 = \omega^2\mu_t\varepsilon_t$

and $\underline{\bar{E}}_t = \hat{y}\underline{E}_o e^{-jkz + jk_{tx}x} = \hat{y}\underline{E}_o e^{-jk_z z + \alpha x}$

L7-8