6.014 Lecture 10: Introduction to Circuits

A.   Kirchoff's Laws

As suggested in Figure L10-1A, circuits are usually analyzed in terms of lumped elements (or branches) that are connected together at nodes to form a two- or three-dimensional circuits, these can be characterized by the voltages at each node or by currents in a set of loops. Although circuit analysis is often based in part on Kirchoff's laws, these laws are imperfect due to electromagnetic effects. For example, Kirchoff's voltage law (KVL) says that the voltage drops V_i associated with each lumped element around any loop must sum to zero, i.e.:

$$\Sigma_i V_i = 0 \tag{1}$$

This follows from Faraday's law, which in differential and integral form is:

$$\nabla \times \overline{E} = -\partial \overline{B}/\partial t \tag{2}$$

$$\int_c \overline{E} \bullet d\overline{s} = -\int_A (\partial \overline{B}/\partial t) \bullet d\overline{a} \tag{3}$$

where the variables are defined in Figure L10-1B. If $\partial \overline{B}/\partial t = 0$ within the loop, then the integral around the loop (around the contour c) is zero and KVL applies. But we know that in any physical circuit the current flowing around a loop is typically non-zero and time-varying, so $\partial \overline{B}/\partial t \neq 0$ and KVL does not apply! We resolve this problem by distinguishing physical from mathematical circuits, and by lumping into separate elements the magnetic energy storage and voltage drops associated with these magnetic fields through the loop. One small inductor might be added in series per branch to form an equivalent lumped element circuit for which KVL applies more precisely, as suggested in Figure L10-2. Normally these parasitic inductances are small and can be neglected.

In a similar fashion Kirchoff's current law (KCL) follows from Ampere's law, which in differential and integral form is:

$$\nabla \times \overline{H} = \overline{J} + \partial \overline{D}/\partial t \tag{4}$$

$$\int_S (\overline{J} + \partial \overline{D}/\partial t) \bullet d\overline{a} = \Sigma_i I_i \tag{5}$$

where the surface S over which the current density $\overline{J}$ is integrated encloses the node of interest and I_i is the current entering that node along branch i. Only if $\int_S \partial \overline{D}/\partial t = 0$ for a given node will the sum of currents into that node be zero so that KCL applies for that node:

$$\Sigma_i I_i = 0 \tag{6}$$

However, if the voltage at a node varies (the only interesting case), then it will produce radial electric fields $\partial \overline{D}/\partial t \neq 0$ so that, in general, $\Sigma_i I_i \neq 0$. Again, the solution is to augment any mathematical circuit with lumped-element capacitors in parallel with each branch to account for this stored electrical energy. Normally these parasitic capacitances are so small and can also be neglected. If both parasitic capacitances and inductances are required for a branch, then the question arises whether they are connected in series or parallel. At this point, or when radiation occurs, distributed circuit models typically must replace simple lumped-element circuits. Later some simple distributed circuits will be analyzed.

B.   Solving circuit problems

Figure L10-4A illustrates a simple circuit with b = 12 branches, p = 6 loops, and n = 7 nodes. A set of loops is unique if each loop circles only one "hole" in the topology and if no additional loops are added once every branch in the circuit is incorporated in at least one loop. It is easy to show that the number b of branches in a circuit is:

$$b = n + p - 1 \tag{7}$$

by considering the simplest possible circuit, which has one node and one branch, as illustrated in Figure L10-4B. As we add either nodes or branches to this circuit, Equation (7) is still obeyed. If we add voltage or current sources to the circuit, they too become branches.

We can see that the total number of unknowns in a circuit is 2b because each branch has both a voltage and a current which we may wish to learn. The number of equations is also b + n + p –1 = 2b, where b in this expression corresponds to the number of voltage/current relations for the branches, n-1 is the number of independent KCL equations, and p is the number of loops and KVL equations. Therefore, since the numbers of unknowns and linear equations match, we may solve them. The equations are linear because Maxwell's equations are linear for R, L, C circuits.

C.   Parallel-plate capacitors

Static electric fields between parallel metal plates such as those in Figure L10-6 are constrained by Maxwell's equations:

$$\nabla \times \overline{E} = 0, \qquad \nabla \bullet \overline{E} = \rho/\varepsilon \tag{8}$$

If there is no free charge density ρ between the plates, then $\nabla \bullet \overline{E} = 0$. Boundary conditions at a perfect conductor are:

$$\overline{n} \bullet \overline{E} = \rho_s/\varepsilon, \qquad \overline{E}_{//} = 0 \tag{9}$$

Since $\overline{E}_{//} = 0$, any $\overline{E}$ must be perpendicular to the conducting sheets, and since the fields must also be both curl-free and divergence-free, the fields must be uniform. Therefore $\overline{E}$ = $\mathbf{y}E_o$ has the correct form, where $\overline{E}$ points away from the positively charged surface, which has charge density $\rho_s = \varepsilon E_o$ [Cm^{-2}]. Therefore, if the plates have area A, the total charge on the capacitor $Q = \rho_s A = \varepsilon E_o A$ Coulombs.

The voltage V across the plates can be found by integrating the electric field from the positive to the negative terminal, so $V = E_o d$.

We define the capacitance C of such an open structure, independent of shape, to be:

$$C = Q/V \quad \text{[Farads]} \tag{10}$$

Here, $C = \varepsilon E_o A / E_o d = \varepsilon A/d$.

Although $Q = CV$ characterizes capacitors in a simple way, it is usually more relevant to relate i(t) to v(t). Since $q(t) = \int_{-\infty}^{t} i(t)\, dt = C\, v(t)$, it follows that:

$$v(t) = (1/C) \int_{-\infty}^{t} i(t)\, dt \tag{11}$$

which is a familiar and useful relation.

Figure L10-9A shows how two capacitors in parallel (C_1 and C_2) are equivalent to a single capacitor C_{eq} having the equivalent charge $Q_{eq} = Q_1 + Q_2$. Therefore two capacitors in parallel have a greater total capacitance of:

$$C_{eq} = Q_{eq}/V = (Q_1 + Q_2)/V = C_1 + C_2 \tag{12}$$

Figure L10-9B shows how two capacitors in series have the same charge Q on each and have a total voltage V across the pair, where V is the sum of the voltages $V_1 = Q/C_1$ and $V_2 = Q/V_2$. Therefore, $1/C_{eq} = V/Q = (V_1 + V_2)/Q = 1/C_1 + 1/C_2$, and two capacitors in series have a smaller total capacitance of:

$$C_{eq} = 1/(C_1^{-1} + C_2^{-1}) \tag{13}$$

D.   Parallel-plate resistors

The same parallel-plate geometry can be used to form resistors R using a medium of conductivity σ [Siemens m^{-1}] bounded by plates of area A and separation d, as suggested in Figure L10-10. Since $\overline{J} = \sigma\, \overline{E}$ [a m^{2}] and $I = AJ = A\sigma E_o = A\sigma V/d = V/R$ amperes, it follows that:

$$R = d/\sigma A \quad \text{[ohms]} \tag{14}$$

The same logic as for capacitors applies when two resistors are placed in series or parallel. When they are in series, the two currents are the same and the two voltages add, whereas if they are in parallel, the two voltages are the same and the currents add. This leads to the expressions:

$$R_{eq} = R_1 + R_2 \text{ ohms } \text{[resistors in series]} \tag{15}$$

$$R_{eq} = 1(R_1^{-1} + R_2^{-1}) \text{ ohms } \text{[resistors in parallel]} \tag{16}$$

E.   Charge relaxation and RC circuits

When a resistor and capacitor are connected as shown in Figure L10-11, any charge of the capacitor will tend to discharge through the resistor, resulting in a decaying voltage as a function of time. We can solve for this voltage v(t) using:

$$q(t) = Cv(t) = \int I(t) \, dt, \quad \text{and} \tag{17}$$

$$v(t) = -i(t)R = -RC \, dv(t)/dt \tag{18}$$

where the direction of i(t) in (18) flows into the positive terminal of the capacitor. The differential equation (18) states that v(t) equals its own first derivative times a constant. Only exponentials have that property, so we guess the solution is:

$$v(t) = v_o e^{-t/\tau} \tag{19}$$

where v_o is an initial condition. Substitution of (19) into (18) yields

$$v_o e^{-t/\tau} = (-RC)(-1/\tau) \, v_o e^{-t/\tau} \tag{20}$$

which is satisfied if $\tau = RC$ seconds.

If we consider the parallel-plate resistor of Figure L10-10, $RC = (d/A\sigma)(\varepsilon A/d) = \varepsilon/\sigma$ seconds, which is the "relaxation time constant" of the medium ε, σ. It is easy to show that this time constant is not a function of the geometry of the plates or initial charge distribution within the uniform medium; the charges, electric fields, and voltages retain their initial distribution, but all decay exponentially from that distribution with time constant $\tau = RC$ seconds.

These two basic elements (R,C) plus inductance and inductors to be considered next suffice to build most passive linear lumped-element and distributed circuits.