6.014 Lecture 11: Inductors and Transformers

A. Inductors

All circuits carry currents that necessarily produce magnetic fields and store magnetic energy. Thus every wire and circuit element generally has some inductance that may influence circuit behavior, particularly at higher frequencies. When two circuit branches share magnetic fields, each will typically induce voltage in the other, thus *coupling* the branches. If this coupling is substantial, the two branches act as a *transformer*.

Figure L11-1 illustrates two circuit elements connected by parallel conducting plates, which approximate a printed circuit wire passing over a conducting ground plane in an integrated or printed circuit. The currents i(t) are equal and opposite. The plates have width W and separation d<<W. Ampere's law enables us to find the magnetic fields and the inductance of this structure per unit length, after which we can discover the nature of inductance. Ampere's law in differential and integral form is:

$$\nabla \times \overline{H} = \overline{J} + \partial \overline{D}/\partial t, \qquad \int_c \overline{H} \bullet d\overline{s} = \int_A (\overline{J} + \partial \overline{D}/\partial t) \bullet d\overline{a} \qquad (1)$$

In the quasistatic limit $\partial/\partial t$ can be ignored and (1) relates the magnetic field $\overline{H}$ to the current density $\overline{J}$. The integral of $\overline{H}$ around the contour C is thus related to the total current I flowing through that contour.

Referring to Figure L11-1, if the contour circles both plates, the total current i(t) is zero because the currents in the two plates are assumed to be equal and opposite. If the contour circles only one plate, then the integral of $\overline{H}(t)$ equals i(t). To proceed, we assume W>>d so that the contributions to the integral of the *fringing fields* at the plate edges can be neglected. We then see that $\overline{H}$ outside the two plates is generally zero because otherwise $\overline{H}$ above and below the plates must point in the same direction, whereas the symmetry of the problem suggests no preferred direction. Furthermore, if $\overline{H}_{outside} \neq 0$ there would be no unique solution to $\overline{H}$ between the plates; consider two contours, one circling the upper plate and one circling the lower plate, but sharing the same path between the plates.[1]

Since $\overline{H}_{outside} \cong 0$, the contour integral (1) yields $\overline{H}_{between} W = -\hat{y}i$, or

$$\overline{H}_{between} = -\hat{y}i/W \quad (2)$$

The inductance of such a parallel-plate structure can be understood by short-circuiting one end at z = 0, as illustrated in sideview in Figure L11-2, and then computing

[1] The neglected fringing fields are antisymmetric and therefore do not contribute to these integrals around symmetric contours.

the electric field $\overline{E}(t,z)$ that must result from $\overline{H}(t)$. Assume the device has plate separation d (now in the y direction), width W, and length D (in the z direction), and has a voltage v(t) across its terminals. The current i(t) in the top plate flows to the right, and the resulting $\overline{H}(t)$ points into the paper.

The electric field $\overline{E}(t,z)$ follows from Faraday's Law, which in differential and integral form is:

$$\nabla \times \overline{E} = -\partial \overline{B}/\partial t \quad \Rightarrow \quad \int_c \overline{E} \bullet d\overline{s} = -\int_A \mu(\partial \overline{H}/\partial t) \bullet d\overline{a} \tag{3}$$

where we use the contour C and cross-sectional area A illustrated in the y-z plane in Figure L11-2. These integrals are trivial to evaluate since $\overline{E}$ inside the perfect conductors is zero, and $H(t) = i(t)/W$ is uniform over the area $A = zd$ [m^2]. Thus the integral form of Faraday's law yields:

$$E_y(z,t)d = -(\mu zd/W)\partial i(t)/\partial t, \text{ and} \qquad E_y(z,t) = -(\mu z/W)\partial i(t)/\partial t \tag{4}$$

The voltage v(t) across the inductor (where z = D) follows from simple integration of $\overline{E}$ (using (4)) from plate 1 to plate 2 (the upper plate):

$$v(t) = \int_1^2 \overline{E} \bullet d\overline{s} = -E_y d = (\mu Dd/W)(\partial i(t)/\partial t) \tag{5}$$

$$v(t) = L\, di(t)/dt \tag{6}$$

where the inductance here is:

$$L = \mu Dd/W \text{ [Henries]} \tag{7}$$

Note that the voltage between these two parallel plates varies with z, as seen from (4) and (5). Thus we have two perfect conductors that have different voltages between them, depending on z. This violates Kirchoff's Voltage Law because of the time-varying magnetic field $\overline{H}$ that threads the loop integral around which we test KVL.

In practice we often want more inductance than is readily supplied using (7), so we modify the structure as suggested in Figure L11-3; we convert the single-turn loop into an N-turn coil by slicing it into wires of width $W_i = W/N$. Equations (6) and (7) then yield the voltage v(t) across one of these turns, which is now N times greater for given i(t) because W is N times smaller. The total voltage across N turns in series is another factor of N times greater. Thus the total voltage across the N-turn coil is:

$$v(t) = L\, di(t)/dt, \text{ where} \quad L = N^2\mu Dd/W = N^2\mu A/W \tag{8}$$

where area $A = Dd$; thus L is N^2 times its previous value.

The magnetic energy density within this inductor L is:

$$W_m = \mu|\overline{H}(t)|^2/2 \quad [\text{J m}^{-3}] \tag{9}$$

Which corresponds to total stored magnetic energy of

$$w_m = \mu AW|\overline{H}(t)|^2/2 = \mu A(Ni)^2/2W \text{ [J]} \tag{10}$$

where H = Ni/W. Combining (8) and (10) yields the useful result:

$$w_m = Li^2(t)/2 \text{ [J]} \tag{11}$$

Inductors generally have some resistance R, which can be readily determined. If we construct our inductors from slabs with conductance σ [Sm^{-1}][2], length D, thickness δ, and cross-sectional area A = δW, then the resistance along the full length of the slab is D/σA [ohms]. Since the length of a single turn is 2(D+d), the total resistance of an N-turn inductor is 2N(D+d)/σA [ohms]. It is, of course, much more important to understand how such values are derived than to memorize any answers.

Should we have an RL circuit as illustrated in Figure L11-4 which has an initial current i(t=0) = I_o, then it is easy to show that i(t) = $I_o e^{-t/\tau}$, where the time constant τ = L/R seconds. For our N-turn inductor we can substitute our values for L and R to yield:

$$\tau = L/R = (N^2\mu Dd/W)/(2N^2(D+d)/\sigma A) \cong \mu d\delta\sigma/2 \text{ [s]} \tag{12}$$

where D>>d and A = (W/N)δ. Thus long time constants τ are achieved by maximizing μ, d^2, and σ, since δ < ~d; this can lead to large massive structures.

B. Transformers

Figure L11-5 illustrates a *solenoidal* (cylindrical) *transformer* comprising two coils wound about the same air-filled cylinder of cross-sectional area A and length W (we assume A and W are the same for both coils). To determine the behavior of the transformer we use the integral form of Faraday's law:

$$\int_c \overline{E} \bullet d\overline{s} = -\int_A \mu_o(\partial \overline{H}/\partial t)\bullet d\overline{a} \tag{13}$$

If we compute the contour integral (13) around one turn of either coil we obtain the same answer, which is μHA, the *magnetic flux* linked by one turn. Therefore the total voltage induced in either coil by the same changing magnetic flux is proportional to its number of turns. This total voltage induced across coil 2 is therefore N_2/N_1 times the voltage across coil 1, where N_2/N_1 is called the *transformer turns ratio* and can be greater or less than unity. If the flux coupling between the two coils is imperfect, then the output voltage is correspondingly reduced. If the wires have resistance, that can alter these voltages in proportion to the currents.

[2] The units of conductance are Siemens m^{-1}, where Siemens are the reciprocal of ohms.

Many transformers have coils wound on iron cores rather than around air, partly in order to reduce flux leakage. Consider the boundary between air and a high-permeability material, as illustrated in Figure L11-6. The boundary conditions are that $\overline{H}_{//}$ and $\overline{B}_{\perp}$ are continuous across any interface. Since $\overline{B} = \mu\ \overline{H}$ in the permeable core and $\overline{B} = \mu_o\ \overline{H}$ in air, where $\mu/\mu_o >> 1$, and since $\overline{H}_{//}$ are equal on both sides of the boundary, therefore $\overline{B}_{//}$ differs by the large factor μ/μ_o. In contrast, $\overline{B}_{\perp}$ is the same on both sides. Therefore, as shown in Figure L11-6, we see that $\overline{B}_2$ in air is nearly perpendicular to the boundary, while $\overline{B}_1$ inside is nearly parallel and therefore largely trapped there, even if that boundary curves. Figure L11-7 shows how $\overline{B}$ can be trapped inside a toroid so that coils can be placed anywhere around its perimeter and still be well coupled since the magnetic flux Λ is approximately constant around the loop, where

$$\Lambda = \int_A \overline{B} \bullet d\,\overline{a} \qquad (14)$$

Note the polarity of the output voltage $v_2(t)$ relative to $v_1(t)$ for the given directions in which the coils in Figure L11-6 are wound. The polarity of $v_2(t)$ relative to $\partial\overline{B}/\partial t$ is governed by (13) and that of $\overline{B}$ relative to $i_1(t)$ is governed by (1).

C. Toroidal Inductors

A toroidal inductor such as that illustrated in Figure L11-8 has inductance L, which is related to the stored magnetic energy by (9) and (11):

$$w_m = Li^2(t)/2 = \int_V (\mu|\overline{H}(t)|^2/2)\,dv \ \ [J] \qquad (15)$$

Finding $\overline{H}(t)$ is easier if the toroid has constant cross-section A, as illustrated in Figure L11-8. From Ampere's law we learn that the integral of $\overline{H}$ around the $2\pi R$ circumference of the toroid is:

$$\int_c \overline{H} \bullet d\,\overline{s} \cong 2\pi RH \cong Ni \qquad (16)$$

where the only linked current is i(t) flowing through the N turns of wire threading the toroid. Equation (16) yields $H \cong Ni/2\pi R$ and (15) relates $\overline{H}$ to w_m and L. Therefore the inductance L of such a toroid is:

$$L = \mu i^{-2}\int_V (Ni/2\pi R)^2 dv \cong \mu(N/2\pi R)^2 2\pi RA = \mu NA/2\pi R \ \ \text{[Henries]} \qquad (17)$$

The inductance is proportional to μ, N, and A, but declines as R increases. The most compact toroids are therefore fat with almost no hole in the middle; the hole size is determined by N and wire diameter.

The inductance of a toroid is strongly affected if even a small gap of width d exists in the magnetic path, as shown in Figure L11-9. To compute the total magnetic energy w_m using (15), the integral $\int_V$ must include all space; the magnetic energy stored in the small gap can then easily dominate. Since $\overline{B}$ is continuous across the gap, $\mu H_\mu = \mu_o H_{\mu_o}$ and $H_{\mu_o} >> H_\mu$. Equation (16), when integrated around a contour C that includes the gap, yields:

$$|\overline{H}_\mu|(2\pi R - d) + |\overline{H}_{\mu o}|d \cong N\,i(t) \cong |\overline{H}_{\mu o}|d \qquad (18)$$

which occurs for sufficiently large values of (μ/μ_o) and modest values for R/d. Thus most non-zero gaps dominate the inductance because $|\overline{H}|$ and w_m are relatively so large there. The approximate inductance L then follows from equations (15) and (18):

$$L \cong \mu_o Ad(Ni/d)^2/i^2(t) = \mu_o AN^2/d \;\; \text{[Henries]} \qquad (19)$$

Comparing (17) and (19) we see that the gap reduces L by a factor of μ/μ_o, but gains a factor of $N2\pi R/d$, which can be made very large. Equation (19) explains how small air gaps in magnetic motors control motor inductance, as discussed further later.