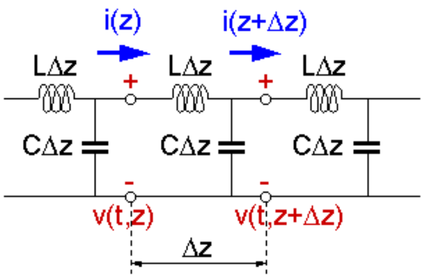
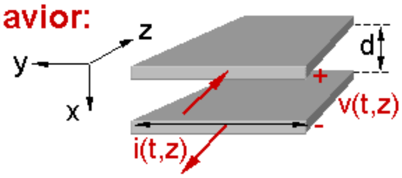
# MICROWAVE CIRCUITS

## Printed Circuits Exhibit R,L,C Behavior:

Equivalent TEM line circuit:




"TEM" $\Rightarrow \bar{E} \bullet \hat{z} = \bar{H} \bullet \hat{z} = 0$

**Difference Equations**

$$v(z+\Delta z) - v(z) = -L\Delta z\, di(z)/dt$$

$$i(z+\Delta z) - i(z) = -C\,\Delta z dv(z)/dt$$

Limit as $\Delta z \to 0$:

$$dv/dz = -L\, di/dt$$

$$di/dz = -C\, dv/dt$$

Let $v(z,t) = R_e\{\underline{V}e^{j\omega t}\}$:

$$d\underline{V}(z)/dz = -j\omega L\underline{I}(z)$$

$$d\underline{I}(z)/dz = -j\omega C\underline{V}(z) \Rightarrow$$

**Wave Equation**

$$d^2\underline{V}(z)/dz^2 + \omega^2 LC\underline{V}(z) = 0$$

L15-1

# TEM SINUSOIDAL STEADY STATE EQUATIONS

**Wave Equation:** $d^2\underline{V}(z)/dz^2 + \omega LC\underline{V}(z) = 0$

**Voltage Solution:** $\underline{V}(z) = \underline{V}_+ e^{-jkz} + \underline{V}_- e^{jkz}$

Test solution: $[(-jk)^2\underline{V}_+ e^{-jkz} + (jk)^2\underline{V}_- e^{jkz}] + \omega^2 LC[\underline{V}_+ e^{-jkz} + \underline{V}_- e^{jkz}] = 0$

Passes test iff: $k^2 = \omega^2 LC$

**Current $\underline{I}(z)$:**

Since: $\partial\underline{V}(z)/\partial z = -j\omega L\ \underline{I}(z)$

Therefore $I(z) = (1/j\omega L)jk(\underline{V}_+ e^{-jkz} - \underline{V}_- e^{jkz})$
$= Y_o(\underline{V}_+ e^{-jkz} - \underline{V}_- e^{jkz})$

[Characteristic admittance $Y_o = k/\omega L = \omega(LC)^{0.5}/\omega L = (C/L)^{0.5} = 1/Z_o$]

**Transmission Line Equations:**

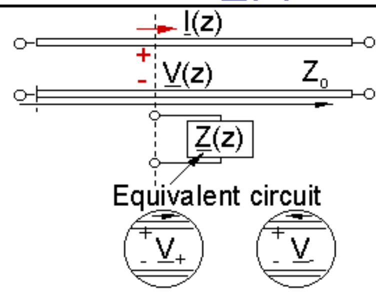
$$\underline{V}(z) = \underline{V}_+ e^{-jkz} + \underline{V}_- e^{jkz}$$

$$\underline{I}(z) = Y_o(\underline{V}_+ e^{-jkz} - \underline{V}_- e^{jkz})$$

L15-2

# COMPLEX LINE IMPEDANCE $\underline{Z}(z)$

## Impedance:




$$\underline{Z}(z) = \underline{V}(z)/\underline{I}(z) = R(z) + jX(z)$$

Resistance    Reactance

$$\underline{Z}(z) = \underline{V}(z)/\underline{I}(z) = \frac{Z_o\left(\underline{V}_+ e^{-jkz} + \underline{V}_- e^{jkz}\right)}{\underline{V}_+ e^{-jkz} - \underline{V}_- e^{jkz}} = Z_o \frac{1+\underline{\Gamma}(z)}{1-\underline{\Gamma}(z)}$$

## Complex Reflection Coefficient $\underline{\Gamma}(z)$:

$$\underline{\Gamma}(z) = \underline{V}_- e^{+jkz} / \underline{V}_+ e^{-jkz} = \underline{\Gamma}_L e^{2jkz} \text{ where } \underline{\Gamma}_L = \underline{\Gamma}(z=0) = \underline{V}_- / \underline{V}_+$$

**Examples:** $\underline{\Gamma} = 0 \Rightarrow \underline{Z}(z) = Z_o$  $\underline{\Gamma} = +1 \Rightarrow \underline{Z} = \infty$  $\underline{\Gamma} = -1 \Rightarrow \underline{Z} = 0$

L15-3

# GENERAL EXPRESSIONS FOR Z(z)

## Complex Reflection Coefficient $\underline{\Gamma}(z)$:

Since $\underline{Z}(z)/Z_o = [1+\underline{\Gamma}(z)]/[1-\underline{\Gamma}(z)] = \underline{Z}_n(z)$

Therefore: $\underline{\Gamma}(z) = [\underline{Z}(z) - Z_o]/[\underline{Z}(z) + Z_o]$

## $\underline{Z}(z)$ as a Function of $Z_L$, $Z_o$, k, and z:

Substituting: $\underline{\Gamma}_L(z) = [\underline{Z}_L - Z_o]/[\underline{Z}_L + Z_o]$

Into:
$$\underline{Z}(z) = \underline{V}(z)/\underline{I}(z) = Z_o \frac{\underline{V}_+ e^{-jkz} + \underline{V}_- e^{jkz}}{\underline{V}_+ e^{-jkz} - \underline{V}_- e^{jkz}}$$
$$= Z_o\left[\left(e^{-jkz} + \underline{\Gamma}_L e^{+jkz}\right)/\left(e^{-jkz} - \underline{\Gamma}_L e^{+jkz}\right)\right]$$

Yields:
$$\underline{Z}(z) = Z_o \frac{(\underline{Z}_L + Z_o)e^{-jkz} + (\underline{Z}_L - Z_o)e^{jkz}}{(\underline{Z}_L + Z_o)e^{-jkz} - (\underline{Z}_L - Z_o)e^{jkz}}$$
$$= Z_o \frac{\underline{Z}_L \cos kz - jZ_o \sin kz}{-j\underline{Z}_L \sin kz + Z_o \cos kz}$$

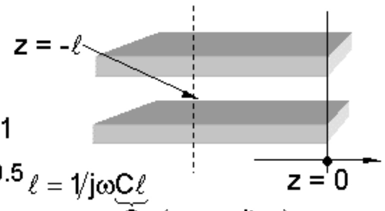
Therefore:
$$\underline{Z}(z) = Z_o \frac{\underline{Z}_L - jZ_o \tan kz}{Z_o - j\underline{Z}_L \tan kz}$$

L15-4

# EXAMPLES OF Z(z) TRANSFORMATIONS

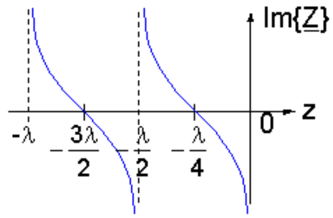
**Transformation Equation:**
$$\underline{Z}(z) = Z_o \frac{\underline{Z}_L - jZ_o \tan kz}{Z_o - j\underline{Z}_L \tan kz}$$

**Example—Open Circuit, $Z_L = \infty$:**




$$\underline{Z}(-\ell) = -jZ_o \cot k\ell = -jZ_o/k\ell \quad \text{for } k\ell << 1$$
$$= -j(L/C)^{0.5}/\omega(LC)^{0.5}\ell = 1/j\omega \underbrace{C\ell}_{C_o}$$
$C_o$ (capacitor)

$= 0$ when $z = -\lambda/4, -3\lambda/4, \ldots$
$= \infty$ when $z = 0, -\lambda/2, \ldots$




In general: $-j\infty < \underline{Z}(-\ell) < +j\infty$

(ANY capacitance or inductance at a SINGLE frequency)

L15-5

# MORE EXAMPLES OF Z(z) TRANSFORMATIONS

**Example—Inductive Load, $Z_L = j\omega L_o$ for $z = -\lambda/4$:**

Recall: $\underline{Z}(z) = Z_o \dfrac{\underline{Z}_L - jZ_o \tan kz}{Z_o - j\underline{Z}_L \tan kz}$

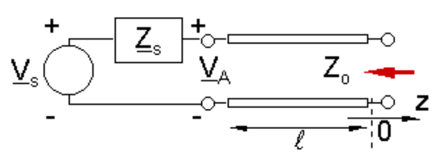
Since: $kz = -k\ell = (2\pi/\lambda)/(\lambda/4) = -\pi/2$, and $\tan(-k\ell) = -\infty$

Therefore: $\underline{Z}(z) = Z_o{}^2/\underline{Z}_L = (L/C)/(j\omega L_o) = 1/(j\omega CL_o/L) = 1/j\omega C_o$

Note: $\underline{Z}(z) = 1/j\omega L_o$ if $\ell = \lambda/2, \lambda, \ldots (\tan-(2\pi/\lambda)\lambda = 0)$

**Example—Transformation of Source Impedances:**

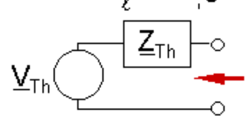
$$\underline{Z}_{Th} = Z_o \frac{\underline{Z}_s + jZ_o \tan k\ell}{Z_o - j\underline{Z}_s \tan k\ell}$$




$\underline{V}_{Th} = 2\underline{V}_+ = ?$

$\underline{V}_A = \underline{V}_s \underline{Z}_A/(\underline{Z}_s + \underline{Z}_A)$ where $\underline{Z}_A = jZ_o \cot k\ell$

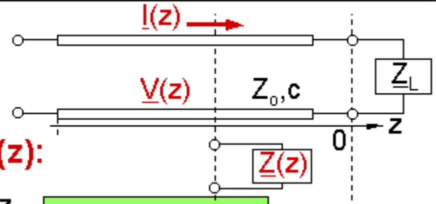
$= \underline{V}_+ (e^{jk\ell} + e^{-jk\ell}) = 2\underline{V}_+ \cos k\ell$




Therefore: $\underline{V}_{Th} = \underline{V}_s \underline{Z}_A / [(\underline{Z}_s + \underline{Z}_A)\cos k\ell]$

L15-6

# ALTERNATE APPROACH TO FINDING Z(z)




## Complex Reflection Coefficient $\underline{\Gamma}(z)$:

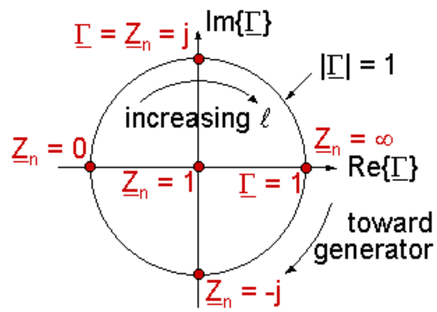
1) $\underline{Z}(z) = \underline{V}(z)/\underline{I}(z) = Z_o \dfrac{\underline{V}_+ e^{-jkz} + \underline{V}_- e^{jkz}}{\underline{V}_+ e^{-jkz} - \underline{V}_- e^{jkz}} = Z_o \dfrac{1+\underline{\Gamma}(z)}{1-\underline{\Gamma}(z)} = \underline{Z}(z)$

2) $\underline{\Gamma}(z) = \underline{V}_- e^{+jkz} / \underline{V}_+ e^{-jkz} = \underline{\Gamma}_L e^{2jkz} = \underline{\Gamma}(z)$ where $\underline{\Gamma}_L = \underline{\Gamma}(z=0) = \underline{V}_-/\underline{V}_+$

3) $\underline{\Gamma}_L(z) = [\underline{Z}_L - Z_o]/[\underline{Z}_L + Z_o]$

## $\Gamma$-Plane Solution Method:

$\underline{Z}_L \Leftrightarrow \underline{\Gamma}_L \Leftrightarrow \underline{\Gamma}(z) \Leftrightarrow \underline{Z}(z)$
(3) (2) (1)

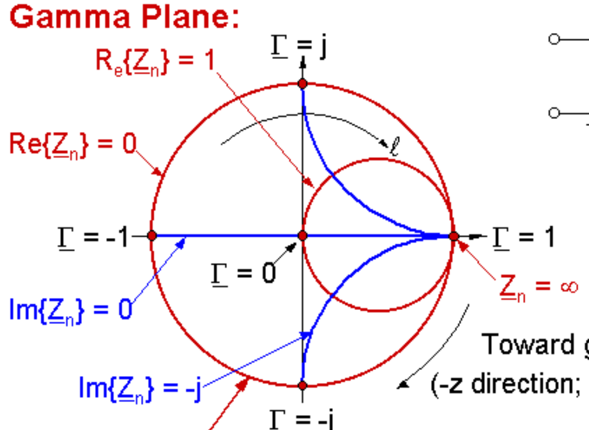


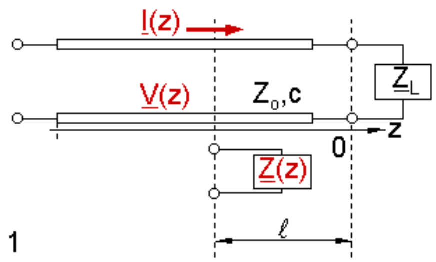

Recall: $\underline{Z}_n = \underline{Z}/Z_o$

L15-7

# GAMMA PLANE ⇒ SMITH CHART

## Gamma Plane:

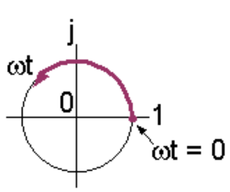



$\underline{Z}_n = (1 + \underline{\Gamma})/(1 - \underline{\Gamma})$

Toward generator (larger $\ell$)
(-z direction; $\lambda/2 \Rightarrow$ full rotation)

## Smith Chart:




Thus: $e^{-2jk\ell}$ goes clockwise as $\ell \to \infty$

$\underline{\Gamma}(z) = \underline{\Gamma}_L e^{2jkz} = \underline{\Gamma}_L e^{-2jk\ell}$

$\underline{Z}_L \Leftrightarrow \underline{Z}_{Ln} \Leftrightarrow \underline{\Gamma}_L \Leftrightarrow \underline{\Gamma}(z) \Leftrightarrow \underline{Z}_n(z) \Leftrightarrow \underline{Z}(z)$

Recall: $e^{j\omega t} = \cos\omega t + j\sin\omega t$

L15-8