6.014 Lecture 24:
Acoustics (Review of 6.014 Wave Concepts)

A. Overview

Many of the most interesting properties of electromagnetic waves arise purely from the wave character of electromagnetic waves and therefore are exhibited by other types of waves as well, such as acoustic waves. Therefore acoustics provides a useful vehicle for reviewing many of these wave phenomena from a different perspective.

Acoustics is also an important field in its own right, and impacts the design of microphones, loudspeakers, theaters, hearing aids, products that emit noise, offices and factories, and other products, and underlies natural phenomena such as speech production and hearing in humans and animals, sound propagation in the environment, certain materials failures (songs that shatter glass), atmospheric waves, and others.

Here we begin with the basic equations of acoustics, which lead immediately to the wave equation, dispersion relation, acoustic Poynting theorem, and the velocity of sound c_s. Snell's law, evanescent waves, and the critical angle phenomenon follow. Acoustic antennas can focus energy using parabolic mirrors or arrays, and pipes can convey acoustic waves in multiple discrete modes with cutoff frequencies and waveguide wavelengths that depend on the mode quantum numbers and waveguide dimensions. Such guides can transform acoustic impedances along their lengths, and can be tuned to match various acoustic loads, just like electromagnetic TEM lines or waveguides. Similarly, resonators have discrete modes and resonant frequencies that depend on cavity dimensions and c_s.

B. Basic Acoustic Variables and Equations

Acoustics is the science of compressive waves in solids, liquids, or gases, although usually the waves are propagating in air. Shear waves are important primarily in solids and will not interest us here.

The choice of primary variables is somewhat arbitrary, as is the case for electromagnetics. For example, we generally treated $\overline{E}$ and $\overline{H}$ as the primary variables in Maxwell's equations, while $\overline{D}$ and $\overline{B}$ were often considered secondary and were determined from $\overline{E}$ and $\overline{H}$ using the constitutive relations. Similarly in acoustics we shall somewhat arbitrarily treat velocity and pressure as primary, and density and temperature as secondary.

Unlike electromagnetic variables that behave in an extremely linear way up to energy levels far beyond anything we encounter in daily life, acoustic waves behave non-linearly at modest power levels and are therefore characterized most easily in terms of perturbations in velocity and pressure. That is, the total pressure P and average particle velocity $\overline{U}$ equal their mean values P_o and $\overline{U}_o$ plus acoustic perturbations p and $\overline{u}$:

$$P = P_o + p \tag{1}$$

$$\overline{U} = \overline{U}_o + u \tag{2}$$

where we assume $\overline{U}_o = 0$ (an important assumption that is not always true).

Both $\overline{E}$ and $\overline{H}$ are vectors orthogonal to the direction of wave propagation, while $\overline{u}$ is generally a vector parallel to the direction of propagation and p is a scalar function of position. The parallel between electromagnetics and acoustics is most evident if we ignore the electromagnetic source terms $\overline{J}$ and ρ:

$$\nabla \times \underline{\overline{E}} = -j\omega\mu_o \underline{\overline{H}} \qquad \nabla \times \underline{\overline{H}} = j\omega\varepsilon_o \underline{\overline{E}} \tag{3}$$

The corresponding *acoustic differential equations* follow from Newton's law (f = ma) and the conservation of mass combined with the gas law:

$$\nabla \underline{p} = -j\omega\rho_o \underline{\overline{u}} \tag{4}$$

$$\nabla \bullet \underline{\overline{u}} = -j\omega(\gamma P_o)^{-1}\underline{p} \tag{5}$$

where γ is the adiabatic constant (a number approximating unity) for the gas in question. Equation (4) simply states that the (negative) spatial gradient in pressure, which is a force density [Nm^{-2}], equals the mass density ρ_o [kg m^{-3}] times the acceleration $j\omega\,\overline{u}$. Equation (5) simply states that diverging mass ($\nabla \bullet \underline{\overline{u}}$) is proportional to the rate at which pressure $\underline{p}$ drops ($-j\omega\underline{p}$). Both pairs of equations lead directly to wave equations; (3) leads to (16), and (4-5) lead to (7). These in turn yield the velocity of light $c = (\mu_o\varepsilon_o)^{-0.5}$, which corresponds to the velocity of sound $c_s = (\gamma P_o/\rho_o)^{0.5}$:

$$(\nabla^2 + \omega^2\mu_o\varepsilon_o)\, \underline{\overline{E}} = 0 \tag{6}$$
$$(\nabla^2 + \omega^2(\rho_o/\gamma P_o))\underline{p} = 0 \tag{7}$$

The solution to the wave equation (7) is, naturally enough, a wave:

$$\underline{p}(\overline{r}) = \underline{p}_o e^{-j\overline{k}\bullet\overline{r}} \tag{8}$$

$$k = 2\pi/\lambda = \omega(\rho_o/\gamma P_o)^{0.5} = \omega/c_s. \tag{9}$$

The *phase velocity for an acoustic wave* is:

$$v_p = \omega/k = (\gamma P_o/\rho_o)^{0.5} = c_s \tag{10}$$

and the *group velocity* is:

$$v_g = (dk/d\omega)^{-1} = (\gamma P_o/\rho_o)^{0.5} = c_s \quad (11)$$

which is the same. For example air at standard surface pressure and 0^oC has $\gamma = 1.4$, $\rho_o = 1.29$ kg m^{-3}, and $c_s = 330$ ms^{-1}. Equation (11) is modified slightly in solids and liquids, for which the "springiness" of air (γP_o) is replaced by the bulk modulus K, so that in solids and liquids the *velocity of sound* is:

$$c_s = (K/\rho_o)^{0.5} \quad (12)$$

The velocity of sound is approximately 1500 ms^{-1} in water and ~1500 – 13,000 ms^{-1} in solids. The velocities are greater in the harder materials such as metals and ceramics.

The general solution (8) to the acoustic wave equation (7) reduces to (13) for monochromatic uniform plane waves confined to z-axis propagation:

$$p(z) = \underline{P}_+ e^{-jkz} + \underline{P}_- e^{+jkz} \quad [Nm^{-2}] \quad (13)$$

$$\overline{\underline{u}}_z(z) = \eta_s^{-1}(\underline{P}_+ e^{-jkz} - \underline{P}_- e^{+jkz}) \quad [ms^{-1}] \quad (14)$$

The similarity to electromagnetic plane waves is obvious—$\underline{P}_\pm$ has replaced $\underline{V}_\pm$, and the variables $\underline{V}(z)$ and $\underline{I}(z)$ have been replaced by their counterparts $\underline{p}(z)$ and $\overline{\underline{u}}(z)$. The *characteristic acoustic impedance* of the medium can be found by substituting (13) into (4) for the positive wave:

$$d\underline{p}_+(z)/dz = -j\omega\rho_o\underline{u}_+ = -jkz\underline{p}_+ \quad (15)$$

$$\eta_s = \underline{p}_+(z)/\underline{u}_+(z) = \omega\rho_o/k = \rho_o c_s = (\rho_o\gamma P_o)^{0.5} \quad [Nsm^{-3}] \quad (16)$$

which is analogous to the definition of the characteristic electromagnetic impedance of free space:

$$\eta_o = \underline{E}_+/\underline{H}_+ = \omega\mu_o/k = \mu_o c = (\mu_o/\varepsilon_o)^{0.5} \ [\text{ohms}] \quad (17)$$

The characteristic electromagnetic impedance η_o for air is ~377 ohms, while the acoustic impedance for air at 20^oC is ~425 [Nsm^{-3}], so the dimensions of acoustic impedance are different.

Acoustic power and energy also have analogs to electromagnetics. For a uniform plane wave in the time domain, *acoustic power density*, or intensity, is:

$$I_s(t) = pu = p^2/\eta_s = \eta_s u^2 \ [Wm^{-2}] = [Nm^{-2}][ms^{-1}] = [Nms^{-1}\bullet m^{-2}] \quad (18)$$

For a uniform electromagnetic plane wave in the time domain,

$$I(t) = E(t)H(t) = E^2/\eta_o = \eta_o H^2 \ [Wm^{-2}] = [Vm^{-1}][Am^{-1}] \quad (19)$$

In terms of sinusoidal steady state variables the time average acoustic intensity of a z-directed uniform plane wave is:

$$< \overline{I}_s(t)> = R_e\{\underline{p}\ \underline{\overline{u}}^*\} = \hat{z}\ |\underline{p}|^2/2\eta_s = \eta_s|\ \underline{\overline{u}}|^2/2 \quad [\mathrm{Wm}^{-2}] \tag{20}$$

For example, using (20) we can easily see that a one-watt per square meter acoustic signal at sea level corresponds to a peak pressure $|\underline{p}| = (1\bullet 2\eta_s)^{0.5} \cong (850)^{0.5} \cong 30\ \mathrm{Nm}^{-2}$, and a peak velocity $|\ \underline{\overline{u}}| = |\underline{p}|/\eta_s \cong 0.07\ \mathrm{ms}^{-1}$. The distance δz the air molecules move during one cycle is $\sim u/\omega \cong 1$ micron at 10 kHz. This is loud compared to the threshold of human hearing, which corresponds to molecular movement of ~1 nm, or a few atomic diameters. The ear achieves this sensitivity by clever impedance transformations that maximize δz, and by averaging the movements of an enormous number of molecules.

An *acoustic Poynting Theorem* can be derived from the acoustic differential equations (4-5) in much the same way the electromagnetic theorem was derived from the corresponding electromagnetic differential equations, yielding:

$$\nabla \bullet (\underline{p}\ \underline{\overline{u}}^*)/2 = -2j\omega(\rho_o|\ \underline{\overline{u}}|^2/4 - |\underline{p}|^2/4\gamma P_o) \quad [\mathrm{Wm}^{-3}] \tag{21}$$

where the left-hand side of (21) is the divergence of acoustic radiated power, and the right-hand side corresponds to $(-2j\omega)$ times the difference between kinetic energy density W_k $[\mathrm{Jm}^{-3}]$, which is proportional to u^2, and potential energy density W_p, which is proportional to p^2. In (21) the dissipative term on the right-hand side has been omitted for simplicity.

C. Acoustic Waves at Planar Boundaries

The *acoustic boundary conditions* are simple and obvious. The perpendicular velocity must be continuous across the boundary, by mass conservation, and the pressure p inside and outside the wall surface must balance for that surface to remain stationary. Therefore at a rigid wall the acoustic pressure p and the parallel velocity $u_{//}$ can be anything, and the perpendicular acoustic velocity $u_\perp$ must be zero, because the wall is motionless.

Consider a uniform plane wave, $\underline{p}_o e^{-j\overline{k}\cdot\overline{r}}$, incident upon a planar surface at angle θ_i, as illustrated in Figure L24-5a. In air we have impedance η_{si} and wave number k_i, and below the surface in the transmitting medium t we have η_{st} and k_t. As in the electromagnetic case, we assume there is a reflected wave and a transmitted wave in order to match all boundary conditions. If we match phases at the boundary we see that all three waves must have the same value for k_z, where $k_z = k_j\sin\theta_j = (\omega/c_{sj})\sin\theta_j$, so that $\theta_r = \theta_i$, and:

$$\sin\theta_i/\sin\theta_t = c_{si}/c_{st} \tag{22}$$

which is *Snell's law for acoustics*.

As in the case of electromagnetic waves, if θ_i is greater than some *critical angle* θ_c, then no real value of θ_t can satisfy (22), and the transmitted wave is evanescent and conveys no time-average power away from the boundary. The critical angle can be found from (22) for the case where $\sin\theta_t = 1$:

$$\theta_c = \sin^{-1}(c_{si}/c_{st}) \tag{23}$$

Such *evanescent acoustic waves* and perfect reflection commonly occur over lakes and ocean when cold water cools the lower air below the temperature of the upper air so that the velocity of sound increases with altitude (see Figure L24-5). In this case acoustic signals emitted close to the horizon are incident upon this cold-warm boundary beyond the critical angle and are perfectly reflected. Moreover, if there is a slight concave form to this acoustic mirror, it can even concentrate and thereby amplify the sound enormously. This is why fishermen at sea and beachcombers can sometimes hear each other talking even though they are too far apart to be heard normally.

D. Acoustic Antennas

Most *acoustic antennas* are vibrating surfaces that radiate from one or both sides of that surface. They create an oscillatory p and $\overline{u}$ with the desired waveform. If both sides of the surface are free to radiate, then these two waves are perfectly out of phase and tend to cancel at the listener unless one wave travels farther or is attenuated. Generally an attempt is made to trap one of the two waves in a box.

Once a spherical wave front is generated it can be reflected from a parabolic mirror (see Figure L24-6a) to produce a planar phase front that can be directed in specific directions. Thus the power radiated in any particular direction can generally be greater than it would have been if the radiator were isotropic. Thus the definition of *acoustic antenna gain* (gain over isotropic) is the same as that for other antennas:

$$G_s(\theta,\phi) = I_s(\theta,\phi,r)/(P_t/4\pi r^2) \tag{24}$$

where P_t is the total acoustic power transmitted, and r is the distance to the observer.

A horn is an alternate form of acoustic antenna. In this case the radiated waves propagate inside an *acoustic waveguide* (tube) that gradually expands to better match the acoustic impedance of free space. Almost all orchestral horns, such as the French horn, have exponentially expanding bells that flare out most dramatically at their ends. An exponential taper results from cascading *quarter-wave acoustic transformers,* just as happens to Z_o when cascading quarter-wave TEM transformers (recall the transformer impedance should be the geometric mean of the impedances at each end of the quarter-wave section, i,e, $Z_o = (Z_A Z_B)^{0.5}$.

At the small end of an acoustic horn a small pressure surface typically moves a large distance per cycle because the acoustic resistance of air is so little. The acoustic power P is conserved as the wave propagates along the horn:

$$P = IA = A|\underline{p}|^2/2\eta_s = A\eta_s|\ \overline{\underline{u}}|^2/2 \quad [W] \tag{25}$$

where I is intensity $\{Wm^{-2}]$ and A is the varying cross-sectional area. Thus the perturbational wave velocity u and pressure p in the horn are inversely proportional to the radius of the pipe. Such horns were often used to direct sounds and acoustic power toward listeners (e.g., megaphones) before microphones and loudspeakers were available, and they were used before hearing aids to amplify sounds. Even the outer human ear helps funnel sounds to the eardrum, and the ears of some mammals (e.g. bats, rabbits, and deer) are even more effective.

Acoustic array antennas can provide more complex directionality, as suggested in Figure 24-6b, where two in-phase acoustic monopoles radiate isotropically so as to produce nulls in directions where the two waves arrive perfectly out of phase. Perhaps animal ears are the most sophisticated, for their deliberate and specific structure with multiple ridges focuses sounds differently depending on frequency. That is, the ear exhibits nulls in a direction that is a function of frequency. As a result, if white noise is heard (e.g. the rustling of leaves due to a predator) the brain can determine the direction of the noise in both the horizontal and vertical planes by noting the frequency of the null. The frequency-dependent shadowing of sounds by the head, and the use of two ears further enable people to guess the distance to a white-sound sources within a few feet.

E. Acoustic Waveguides and Resonators

As discussed above in Section C, acoustic waves must be reflected at rigid planar surfaces in order to match phases along the boundary, much as are electromagnetic waves, as suggested in Figure L24-7a. The figure shows the phase fronts for two waves crossing each other, where one is the incident wave, and the other is the reflected wave. The solid lines mark phase fronts of maximum positive velocity, and the dashed lines mark phase fronts of maximum negative velocity, where the velocity $\overline{u}$ of a wave is perpendicular to the wave phase front. At any of the horizontal lines in the figure the vertical components of $\overline{u}$ for the two waves cancel, although the horizontal components add. At any of these loci a hard boundary can be located that reflects the waves. Two such boundaries can trap waves between them.

The acoustic waveguide wavelength λ_g varies with wavelength for two reasons: the free-space wavelength λ_o is varying, and the angle of incidence must vary in order to match the boundary conditions at both the top and bottom of the guide. A *parallel-plate acoustic waveguide* has one *quantum number,* the number of half-wavelengths between the top and bottom plate. For example, the symbol A_m represents an acoustic parallel-plate mode with m half-wavelengths λ_x across the width of the guide, where $\lambda_x = 2\pi/k_x$

and $k_x = k_o\cos\theta_i$; that is, $\lambda_x = \lambda_o/\cos\theta_i$. Similar boundary conditions must be satisfied in a rectangular acoustic waveguide, where the modes are characterized by $A_{m,n}$.

Rectangular resonant cavities must match similar boundary conditions in all three dimensions D_x, D_y, and D_z, so resonator modes are characterized as $A_{m,n,q}$, where the quantum numbers are related to the dimensions as:

$$D_x = m\lambda_x/2,\ D_y = n\lambda_y/2, \text{ and } D_z = q\lambda_z/2, \text{ and:} \tag{26}$$

$$k_i = 2\pi/\lambda_i \quad (i = x, y, \text{ or } z) \tag{27}$$

$$\Sigma_i\, k_i^2 = k_o^2 = 4\pi^2\Sigma_i\, \lambda_i^{-2} \tag{28}$$

This is similar to the modal structure of rectangular electromagnetic resonators. Equation (27) can be expressed in terms of the mode quantum numbers and cavity dimensions by using (26):

$$(\omega/c_s)^2 = k_o^2 = \Sigma_i\, k_i^2 = \Sigma_i\, (2\pi/\lambda_i)^2 = (2\pi)^2[(m/D_x)^2 + (n/D_y)^2 + (q/D_z)^2] \tag{29}$$

Equation (29) can now be solved for the acoustic resonant frequencies of a rectangular resonator:

$$f_{m,n,p} = \omega_{m,n,p}/2\pi = c_s[(m/D_x)^2 + (n/D_y)^2 + (q/D_z)^2] \quad [\text{Hz}] \tag{30}$$

Equation (30) can be interpreted geometrically, as shown in Figure L24-7b where the length of the radius vector is frequency f_{mnp} for the mode characterized by those quantum numbers that mark the axes. The lengths of the three axes follow from (30) and are: mc_s/D_x, nc_s/D_y, and qc_s/D_z. From this simple picture it is easy to see that the number of resonant modes in a rectangular cavity increases as f^3 and with the volume V of the cavity. The density of modes in a cavity [modes Hz^{-1}] increases with the volume of a thin shell of thickness df [Hz], as illustrated; thus this density increases as f^2. The number N of acoustic modes in df Hz thus equals the volume of the shell divided by the volume of a unit cell corresponding to a single mode:

$$N = df\,(\pi f^2/2)/(c_s^3/V) \tag{31}$$

and the number of cavity modes per Hz is therefore $\pi f^2 V/2c_s^2$. Thus, although the human ear can resolve room resonances in small rooms at low frequencies, these resonant frequencies become too numerous and dense to distinguish in larger rooms and higher frequencies.